

# Regularity up to the Crack-Tip for the Mumford-Shah problem.

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## Abstract

We will prove that if  $(u, \Gamma)$  is a minimizer of the functional

$$J(u, \Gamma) = \int_{B_1(0) \setminus \Gamma} |\nabla u|^2 dx + \mathcal{H}^1(\Gamma)$$

and  $\Gamma$  connects  $\partial B_1(0)$  to a point in the interior, then  $\Gamma$  satisfies a point-wise  $C^{2,\alpha}$ -estimate at the crack-tip.

This means that the Mumford-Shah functional satisfies an additional, and previously unknown, Euler-Lagrange condition.

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# 1 Introduction.

The Mumford-Shah functional

$$J(u, \Gamma) = \int_{\Omega \setminus \Gamma} |\nabla u|^2 + \alpha(u(x) - h(x))^2 dx + \beta \mathcal{H}^{n-1}(\Gamma \cap \Omega)$$

was introduced by David Mumford and Jayant Shah in [21] in the context of image processing problems. The idea is to find  $u$  as the “piecewise smooth” approximation of the given raw image data  $h(x)$ .

In this paper we will investigate the regularity of the free discontinuity set of minimizers to the following simplified version of the Mumford-Shah functional

$$J(u, \Gamma) = \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + \mathcal{H}^1(\Gamma \cap \Omega) \quad (1)$$

where  $\Omega \subset \mathbb{R}^2$  is a given set,  $u \in W^{1,2}(\Omega \setminus \Gamma)$ ,  $u|_{\partial\Omega} = g$  and  $\Gamma$  is a one dimensional set that is not apriori determined. In particular, minimizing (1) involves finding a pair  $(u, \Gamma)$  where the function  $u$  is allowed to be discontinuous across  $\Gamma$  but the set  $\Gamma$  can not have too large one dimensional Hausdorff measure  $\mathcal{H}^1$ . It is the balance between the bulk energy  $\int |\nabla u|^2 dx$  and the surface energy  $\mathcal{H}^1$  that makes the problem interesting and challenging. We will call the set  $\Gamma$  the free discontinuity set. Notice that the absence of the term with the image data requires imposing boundary conditions.

It is also important to notice that the concept of the competing bulk and surface energies goes back to British engineer Alan Arnold Griffith (see [18]) whose theory of brittle fracture is based on the balance between gain in surface energy and strain energy release. Our results are more relevant for this interpretation of the minimizers rather than to image processing.

The existence and regularity of the Mumford-Shah minimizers started by pioneering works of Ennio De Giorgi, Michele Carriero, Antonio Leaci, Luigi Ambrosio, Guy David, Alexis Bonnet, Nicola Fusco and Diego Pallara (see [14], [1], [12], [8], [4], [2], [9]). Most of the known results can be found in the following two monographs [3], [13]. We would also like to mention some recent publications, such as [19], [16], [15].

The regularity analysis near the crack-tip is of particular difficulty and interest, since the crack-tip is the only singularity where the bulk energy and the surface energy in a ball  $B_r$  scale of the same order as  $r \rightarrow 0$ . As a result one cannot exploit the domination of the surface term over the bulk term, and the methods known from the theory of minimal surfaces does not apply.

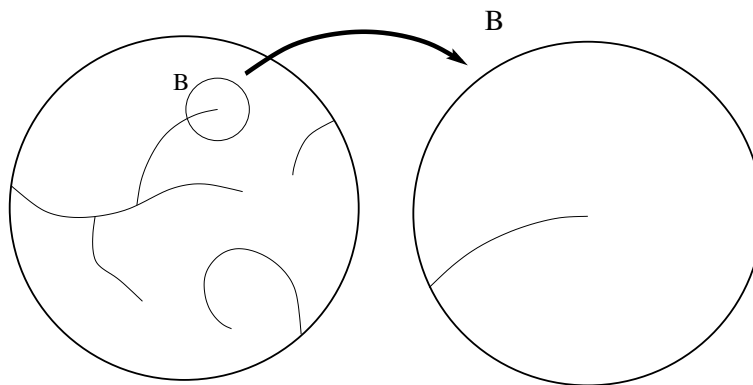
In the paper we apply linearization techniques to derive the full asymptotic of both, analytic and geometric, components of the minimizers near the crack-tip. These classical techniques has been recently successfully exploited by the first author, Henrik Shahgholian and Georg Weiss in regularity analysis of several free boundary problems (see [7], [6]). In Sections 3 and 4 we adapt those free boundary theory methods to free discontinuity context. The exact asymptotic of the minimizers allow to carry out variations of the discontinuity set in the orthogonal direction near the crack-tip and derive a new, previously unknown, Euler-Lagrange condition for the Mumford-Shah functional in Section 7.

We believe that our results are of significant importance for the regularity analysis of the quasi-static crack-propagation model of Gilles A. Francfort and Jean-Jacques Marigo (see [17], [11]). In Section 1.2 we briefly discuss this.

In the next subsection we will describe, in more detail, the problem setting. Then we will briefly summarize background and the relevant known results for the Mumford-Shah problem. We end the introduction with stating our main results.

### 1.1 Problem Setting.

The Mumford-Shah problem consists in finding the pair  $(u, \Gamma_u)$  that minimizes the energy (1) with some prescribed boundary values  $u = g$  on  $\partial\Omega$ . In general, the set  $\Gamma$  can be very complicated (disconnected etc.) as in the left figure below.



**Figure 1:** The geometry of a blow-up.

It has been shown (see [4], [2], [3] and [13]) that the free discontinuity set  $\Gamma_u$  is a  $C^{1,\alpha}$ -graph in some coordinate system around  $\mathcal{H}^1$ -almost every point  $x^0 \in \Gamma_u$ . It is also known that certain singularities exist [13] such as *spider points*, consisting of three arcs meeting at  $120^\circ$  angles in a point, or *crack-tips*, points where the free discontinuity set  $\Gamma_u$  ends at some point. The right figure above shows a typical crack-tip.

Not much is known about the regularity properties of the free discontinuity set close to the crack-tip. In particular, it is not known if the free discontinuity set can spiral around a crack-tip point infinitely many times or if the blow-up of the solution is unique at the crack-tip. Our main goal is to analyze the behavior of  $\Gamma_u$  close to a crack-tip. In this article we will provide an analysis of the crack-tip and exclude spiraling behaviors as well as providing good regularity estimates at the crack-tip.

In order to specify what we mean by a crack-tip we make the following definition.

**Definition 1.1.** *We say that  $(u, \Gamma)$  is  $\epsilon$ -close to a crack-tip if the following holds:*

1.  $(u, \Gamma)$  is a minimizer of (1) in  $B_1(0) (= \Omega)$  with some specified boundary conditions.

2.  $\Gamma$  consists of a connected rectifiable curve that connects the origin to  $\partial B_1(0)$ .

3. And for some  $\lambda \in \mathbb{R}$

$$\left( \int_{B_1 \setminus \Gamma} \left| \nabla \left( u - \lambda \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) \right) \right|^2 dx \right)^{\frac{1}{2}} \leq \epsilon.$$

where  $(r, \phi)$  are the standard polar coordinates of  $\mathbb{R}^2$ .

4.  $u(0, 0) = 0$ .

Let us briefly indicate that this definition is not vacuous and that solutions that are  $\epsilon$ -close to a crack-tip indeed do exist.

It is not absolutely clear what is to be considered to be the right class of minimizers. In [1] L. Ambrosio showed existence of minimizers in the space of special functions of bounded variation SBV, in that case the free discontinuity set  $\Gamma_u$  is considered to be the singular support of the measure  $\nabla u$ . Dal Maso, Morel and Solimini [10] showed existence of minimizers under the restriction that  $\Gamma_u$  should consist of at most  $k$  components. In [10] it was also showed that the minimizer with  $\Gamma_u^k$  consisting of at most  $k$  components will converge to the general minimizer as  $k \rightarrow \infty$ , see [10] for exact statements and details. In any case, the existence of minimizers are already well established. For definiteness we can think of minimizers in the sense of [10].

To see that the assumption 2 in Definition 1.1 is not very restrictive we consider a minimizer constructed in [10] with  $\Gamma_u$  consisting of a finite number of components. Then if  $x^0$  is at a crack-tip we may find a small ball  $B_r(x^0)$  such that  $\Gamma_u$  connects  $\partial B_r(x^0)$  to  $x^0$  and  $\Gamma_u \cap B_r(x^0)$  consists of one component. We may then define

$$u_r(x) = \frac{u(rx + x^0)}{\sqrt{r}}. \quad (2)$$

By scaling invariance of the functional  $u_r$  is a minimizer in  $B_1(0)$ ; thus  $u_r$  satisfies 1 of Definition 1.1. Also, since  $\Gamma_u$  have finitely many components,  $\Gamma_u$  will satisfy 2 of Definition 1.1 if  $r$  is small enough. The geometry of the situation is indicated in Figure 1 where we have tried to depict that in a small ball around a crack-tip the free discontinuity set is a curve connecting the origin to the boundary. The right picture in Figure 1 shows the free discontinuity set of the rescaled function  $u_r$ .

If we let  $r$  be small enough in (2) then 3 will be satisfied by  $u_r$ ; this was shown in [8]. In 3 of Definition 1.1, as well as later in this article, we place the branch-cut of  $r^{1/2} \sin(\phi/2)$  along  $\Gamma_u$ .

Since it is well known that minimizers of the Mumford-Shah functional are  $C^{1/2}(B_1 \setminus \Gamma_u)$  the final assumption in Definition 1.1 can be achieved by adding a constant to  $u$ .

It is therefore clear that we can construct minimizers  $(u, \Gamma_u)$ , as in [10], such that up to a rescaling, as in (2), and an additive constant the solution is  $\epsilon$ -close to a crack-tip at every end point of the set  $\Gamma_u$ . Whether the same is true for the more general class of SBV-minimizers considered in [1] is as far as we know an open question. We will not consider that question here.

**Variations:** Before we continue we need to talk about variations and Euler-Lagrange equations. We have talked about two kinds of minimizers, and for definiteness mentioned that we are interested in the minimizers where  $\Gamma_u$  has finitely many components. In practice we are interested in any kind of minimizer that is  $\epsilon$ -close to a crack-tip that satisfies the three types of variations that we will discuss presently.

The first type of variations are variations in  $u$ . Let  $(u, \Gamma_u)$  be a minimizer in  $\Omega$  and let  $\Sigma \subset \Omega$ . Then for any function  $v \in W^{1,2}(\Sigma)$  such that  $v = 0$  on  $\partial\Sigma \setminus \Gamma_u$  we have that  $(u + \epsilon v, \Gamma)$  is a competitor for minimality in the subset  $\Sigma$ . Extending  $v$  by zero in  $\Omega$  we can calculate

$$0 = \frac{dJ(u + \epsilon v, \Gamma)}{d\epsilon} \Big|_{\epsilon=0} dx = 2 \int_{\Sigma} \nabla v \cdot \nabla u dx = 2 \int_{\partial\Sigma \cap \Gamma} v \frac{\partial u}{\partial \nu} d\mathcal{H}^1.$$

Thus minimizers  $u$  satisfy a Neumann boundary condition on  $\Gamma$ .

The second type of variations are the domain variations. These variations we will only do away from the crack-tip. For any function  $\eta \in C^\infty(\Omega; \mathbb{R}^2)$  with compact support it follows that  $v_\epsilon(x) = u(x + \epsilon\eta(x))$  and  $\Gamma_{v_\epsilon} = \{x; x + \epsilon\eta(x) \in \Gamma_u\}$  is a competitor for minimality. A standard calculation, [3], leads to

$$\begin{aligned} \frac{J(u, \Gamma_u) - J(v_\epsilon, \Gamma_{v_\epsilon})}{\epsilon} &= \int_{\Omega} [|\nabla u|^2 \operatorname{div}(\eta) - 2\langle \nabla u, \nabla \eta \cdot \nabla u \rangle] dx - \\ &\quad - \int_{\Gamma_u} \operatorname{div}_{\Gamma_u} \eta d\mathcal{H}^1, \end{aligned} \quad (3)$$

where  $\operatorname{div}_{\Gamma_u} \eta$  is the tangential divergence on  $\Gamma_u$ . For us, the most important situation will be when  $\Gamma_u$  is a graph of a  $C^1$  function on the support of  $\eta$ . If  $\Gamma_u = \{(x_1, f(x_1)); f \in C^1\}$  and  $\eta = \phi e_2$  then

$$\int_{\Gamma_u} \operatorname{div}_{\Gamma_u} \eta d\mathcal{H}^1 = \int_{\Gamma_u} \left( \frac{f'}{1 + |f'|^2} \frac{\partial \phi}{\partial x_1} + \frac{|f'|^2}{1 + |f'|^2} \frac{\partial \phi}{\partial x_2} \right) d\mathcal{H}^1$$

Choosing  $\eta = \phi(x_1, x_2) \frac{(-f'(x_1), 1)}{\sqrt{1 + |f'|^2}}$  and converting the volume integral in (3) to a boundary integral by means of an integration by parts, assuming that everything is smooth, shows that

$$\int_{\Gamma_u^\pm} \pm |\nabla u|^2 \phi(x_1, f(x_1)) d\mathcal{H}^1 - \int_{\Gamma_u} \frac{\partial}{\partial x_1} \left( \frac{f'(x_1)}{\sqrt{1 + |f'|^2}} \right) \phi(x_1, f(x_1)) d\mathcal{H}^1, \quad (4)$$

where  $\Gamma_u^\pm$  indicates whether we consider the values of  $|\nabla u|$  on the upper or lower part of  $\Gamma_u$ , see [3]. We interpret that as

$$\frac{\partial}{\partial x_1} \left( \frac{f'(x_1)}{\sqrt{1 + |f'|^2}} \right) = [|\nabla u(x_1, f(x_1))|^2]^\pm \quad (5)$$

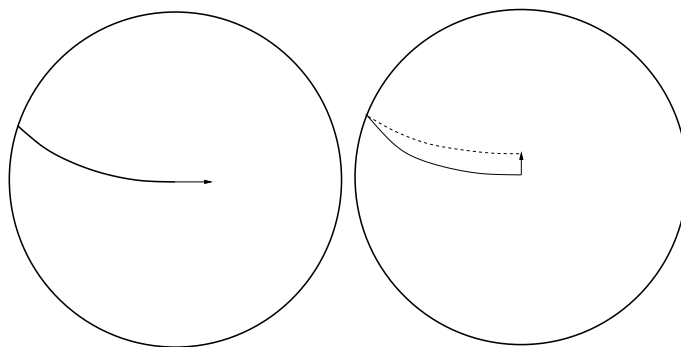
in a weak sense where

$$[|\nabla u|^2]^\pm = |\nabla u(x_1, f(x_1)^+)|^2 - |\nabla u(x_1, f(x_1)^-)|^2.$$

The final type of variation we will do is to change the position of the crack-tip. We assume that  $(u, \Gamma_u)$  is  $\epsilon$ -close to a crack-tip - this includes the assumptions that the crack-tip of  $(u, \Gamma_u)$  is located at the origin and that  $(u, \Gamma_u)$  is a minimizer. By minimality it follows that

$$J(u, \Gamma_u) \leq J(v, \Gamma_v)$$

for any pair  $(v, \Gamma_v)$  where  $\Gamma_v$  is some arc in  $B_1$  and the crack tip of  $\Gamma_v$  is at some point  $x^1 \neq 0$ . This third type of variation has previously been done in the tangential direction of the crack. We will do such tangential variations of the crack-tip in section 2. In section 7 we will make comparisons with functions that has the crack-tip slightly moved in the in the direction orthogonal the crack-tip, see the right figure below. These variations are, to our knowledge, entirely new and needed to show that the curvature vanishes at the crack-tip. These variations require much calculation but are in principle trivial once one has good enough asymptotic information about the solution in a neighborhood of the crack-tip.



**Figure 2.** In the “tangential variations” of the crack-tip we extend the crack as in the left picture. In the “orthogonal variations” of the crack-tip we compare the energy of the crack to the energy of a crack that has the crack-tip moved slightly in the orthogonal direction of the crack tip. This is shown in the right picture with the comparison crack represented by the dashed line.

## 1.2 Background.

In this sub-section we gather some known results and also try to situate the present research within the field. As was already remarked the question of existence of minimizers is already established in different settings [1] or [10]. The next step is usually to prove that the minimizer is more regular than the generic function in the space we minimize in. For the Mumford-Shah problem this means to show two things, that the minimizing function  $u$  is continuous in  $B_1 \setminus \Gamma_u$  and that the free discontinuity set  $\Gamma_u$  is smooth (except at some controllable set of singular points). The more important, and difficult, problem is to establish that  $\Gamma_u$  is smooth. A typical result in this direction is.

**Theorem 1.1.** *There exists a  $\gamma_0 > 0$  such that if  $(u, \Gamma)$  be a minimizing pair of the Mumford-Shah functional and , for some  $\gamma < \gamma_0$ ,*

$$\frac{1}{s} \int_{B_s \setminus \Gamma} |\nabla u|^2 dx < \gamma \quad (6)$$

and

$$\Gamma \subset \{(x_1, x_2); |x_2| < s^2 \gamma\}$$

then  $\Gamma \cap B_{s/2}(0)$  is a  $C^{1,1/4}$ -graph, of a function  $f \in C^{1,1/4}$ ,

$$\Gamma = \{(x_1, f(x_1)); x_1 \in (-s/2, s/2)\}$$

and

$$\|f\|_{C^{0,1}(B_{s/2}(0))} \leq C s^{\frac{1}{4}} \left(1 + \frac{\gamma}{s}\right)^{\frac{1}{2}} \quad (7)$$

for some constant  $C$ .

This theorem is a slightly weaker version of the regularity result in [3] and is close to the state of the art regularity theory for the Mumford-Shah problem in  $\mathbb{R}^2$ . The authors of [3] shows a stronger result in  $\mathbb{R}^n$ , but the above theorem is good enough for our purposes, and easier to formulate.

The first thing to notice about Theorem 1.1 is that the condition (6) excludes the crack-tip. The conclusion of the theorem includes the statement that  $B_{s/2} \setminus \Gamma_u$  is disconnected which isn't true for the crack-tip. The problem is that at a crack-tip  $x^0$

$$u_r(x) = \frac{u(rx + x^0)}{\sqrt{r}} \rightarrow \sqrt{\frac{2}{\pi}} r^{1/2} \sin((\phi + \phi_0)/2) \quad (8)$$

as  $r \rightarrow 0$  through some sub-sequence. This means that, for small  $r$ ,  $u_r$  will not satisfy (6) for small  $\gamma$ .

We would like to remark that the constant  $\sqrt{\frac{2}{\pi}}$  in the limit solution in (8) expresses the right balance between the surface and Dirichlet energy in the functional and is uniquely determined by the functional, see Lemma 2.1.

Also the estimate (7) is not good enough to analyze points close to the crack-tip. In particular, the right hand side in the estimate (7) includes the term  $\gamma$  which measures the size of  $\|\nabla u\|_{L^2}^2$ . However, from the Euler-Lagrange equations (5) we know that the geometry of  $\Gamma_u$  is determined by the difference  $[\|\nabla u\|^2]^\pm$ . If we have symmetry and cancellation in  $[\|\nabla u\|^2]^\pm$  then the curvature of  $\Gamma_u$  might be very small even though  $\|\nabla u\|_{L^2}^2$  is not smaller than  $\gamma$ . At the crack-tip we know that, after a rotation of the coordinate system,

$$u \approx \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2).$$

In particular, we almost have symmetry of the minimizer  $u$  close to the crack-tip. This opens up for a more refined analysis of the regularity of  $\Gamma_u$  close to the crack-tip.

We will indeed use this symmetry effect in our proof - but it is hidden away after the obscure equation (33) in the middle of a long and technical calculation.

There are other regularity proofs for the crack-tip in [8] and [13]. However none of them are using the symmetry close to the crack tip and their proofs does not include the regularity of the free discontinuity sets up to the crack-tip.

Let us finish this introduction by remarking that the regularity of the crack-tip is a very important question for questions relating to crack-propagation in the applied sciences. In particular, [11] showed existence of solutions for Griffith's model of quasi-static propagation in brittle materials. The models involve minimizing  $J(u(x, t), \Gamma_{u(x, t)})$  for each time  $t \in [0, T]$  where  $u(x, t) = g(x, t)$  on  $\partial\Omega$  and under the extra constraint  $\Gamma_{u(x, s)} \subset \Gamma_{u(x, t)}$  for  $s \leq t$ . Their method of analysis is by time discretization. That is, they minimize the problem at discrete times  $t_k = k\delta$  and find solutions to the Mumford-Shah problem  $(u(x, t_k), \Gamma_{u(x, t_k)})$  with the extra condition  $\Gamma_{u(x, t_k)} \subset \Gamma_{u(x, t_{k+1})}$ . By sending  $\delta \rightarrow 0$  they recover a solution to the original problem. In order to analyze this time discretized problem we need to analyze the set free discontinuity set  $\Gamma_{u(x, t_k)} \setminus \Gamma_{u(x, t_{k-1})}$  for each  $k$ . But if  $\delta$  is small then the set  $\Gamma_{u(x, t_k)} \setminus \Gamma_{u(x, t_{k-1})}$  will be close to a crack-tip. It will, in particular, be of great importance to be able to make variations “in the orthogonal direction” as we do in section 7 in order to analyze the growth of a fracture. We plan to address this problem in a future article.

### 1.3 Main Results.

This article has two main results and one technical development, which is strongly related to these results. The first main result is

**Main Result 1.** [SEE COROLLARY 6.1.] *There exists an  $\epsilon_0 > 0$  and a constant  $\alpha_1 \approx 1.1844$  such that for every  $\alpha < \alpha_1 - 1/2$  there exists a constant  $C_\alpha$ , depending on  $\alpha$  but not on  $\epsilon$ , such that if  $(u, \Gamma_u)$  is  $\epsilon$ -close to a crack-tip for some  $\epsilon < \epsilon_0$  then, after possibly rotating the coordinate system*

$$\Gamma_u \subset \{(x_1, x_2); |x_2| \leq C_\alpha \epsilon |x_2|^{1+\alpha}\}.$$

This will be restated as Corollary 6.1 in section 6. We interpret this as a point-wise  $C^{1, \alpha}$  result at the crack-tip. To actually show that  $\Gamma$  is a  $C^{1, \alpha}$  manifold up to the crack-tip would require an argument, similar to the argument in [3] or the main argument here, taking the symmetry into consideration at points close to the crack-tip. We do not expect any difficulties in proving such a result. However, that would add to the length of an already long and technical paper. We also believe that the point-wise regularity at the crack-tip, and the techniques leading up to this result, is of greater importance in order to analyze the quasi-static crack growth.

The proof of the first main result consists of linearizing the Euler-Lagrange equations close to a crack-tip. Solutions of the linearized equations can be expressed as a series of homogeneous solutions. The possible homogeneities of the solutions are given by solutions  $\alpha_k \geq 0$  of the following equation

$$\tan(\alpha\pi) = \frac{2}{\pi} \frac{\alpha}{\alpha^2 - \frac{1}{4}}. \quad (9)$$

A standard flatness improvement argument shows that the solutions of the Euler-Lagrange equations are almost as regular as the least regular homogeneous solution of the linearized problem.



The point-wise  $C^{1,\alpha}$  regularity in Main Result 1 cannot be improved upon without making variations of the crack-tip in the  $x_2$ -direction. One of the main steps in proving Main Result 1 is to determine the asymptotic expansion of the linearized Euler-Lagrange equations at the crack-tip, see Lemma 6.1. Using this explicit expression for the linearized equations we are able to make variations of the crack-tip in the  $x_2$ -direction and derive our second main result. This is done by explicit calculation in section 7. The second main result is.

**Main Result 2.** [SEE THEOREM 7.1.] *There exists an  $\epsilon_0 > 0$  and an  $\alpha_2 \approx 2.099$  such that if  $(u, \Gamma)$  is  $\epsilon$ -close to a crack-tip solution for some  $\epsilon \leq \epsilon_0$  then for every  $\alpha < \alpha_2 - 3/2$  there exists a constant  $C_\alpha$  such that, after possibly rotating the coordinate system,*

$$\Gamma_u \subset \{(x_1, x_2); |x_2| < C_\alpha \epsilon |x_1|^{2+\alpha}, x_1 < 0\}.$$

Here  $C_\alpha$  may depend on  $\alpha$  but not on  $\epsilon < \epsilon_0$ .

We also interpret this as a point-wise  $C^{2,\alpha}$ -regularity result at the crack tip.

**Remark:** The  $C^{2,\alpha}$ -regularity at the crack-tip is a new Euler-Lagrange condition for minimizers to the Mumford-Shah problem. We may minimize the Mumford-Shah energy, say in the setting of [10] with  $\Gamma$  consisting of one component, with an extra constraint that the crack-tip should be at the point  $x^0$ . Then the minimizing function  $u$  is harmonic and  $(u, \Gamma)$  satisfies (5). Before this article it was known that if the crack-tip was free then (8) would be satisfied (possibly after a rotation depending on the sequence  $r_j \rightarrow 0$ ). However, if the crack tip is free then the conclusion of Main Result 2 also have to hold (after a rotation of the coordinates). This last condition is new and dependent on the new variations of the crack-tip in the  $x_2$ -direction. In particular, by making a variation of the crack-tip in the  $x_2$ -direction one can show that the first term in the homogeneous expansion of the limit function in a linearization process will be zero. This implies that the regularity of the solution for a minimizer will be determined by the homogeneity of the second solution  $\alpha_2$  to (9) - which is exactly what Main Result 2 states.

These ideas were announced in [5] and numerically verified in [20].

## 1.4 Notation and Conventions.

We will use several notational conventions in this article. We will freely switch from Cartesian coordinates  $(x_1, x_2)$  to polar coordinates  $(r, \phi)$ . The one dimensional Hausdorff measure will be denoted  $\mathcal{H}^1$ . We will often use  $\Gamma_0 = \{(x_1, 0); x_1 \in (-1, 0)\}$ . The projection operator  $\Pi(u, s)$  is defined in Definition 2.1. In situations when we have some minimizer  $(u, \Gamma_u)$  and consider some other function, almost always  $\lambda r^{1/2} \sin(\phi/2)$ , that needs a branch cut then we assume that the branch cut is along  $\Gamma_u$ . We will also use  $\nu$  for the normal of a given set. At times we will denote the upper and lower normal of  $\Gamma_u$  by  $\nu^\pm$ .

We will at time refer to the space  $H^{1/2}(\partial B_1(0) \setminus \{y\})$  where  $y \in \partial B_1(0)$ . When we do this we mean the space of all traces on  $\partial B_1(0)$  of Sobolev functions  $v \in W^{1,2}(B_1(0) \setminus \Gamma_0)$ . We also remark that this space is equivalent to the space of traces of Sobolev functions  $v \in W^{1,2}(B_1(0) \setminus \Gamma_u)$  for any one dimensional set  $\Gamma_u$  that is  $C^1$  close to  $\partial B_1(0)$ , non-tangential to  $\partial B_1(0)$  and  $\Gamma_u \cap \partial B_1(0) = \{y\}$ . This implies that we can talk about  $H^{1/2}(\partial B_1(0) \setminus \Gamma_u)$  for any minimizer  $(u, \Gamma_u)$  that is  $\epsilon$ -close to a crack-tip.

We will also use  $H^{-1/2}(\Gamma)$ , the dual space to  $H^{1/2}(\Gamma)$ , where  $\Gamma$  is a Lipschitz curve. We will identify a function  $v \in H^{-1/2}(\Gamma)$  with a divergence free vector field  $\eta$ . In particular, if  $w \in H^{1/2}(\Gamma)$  then  $w$  is the trace of some  $W^{1,2}$  function that we still denote  $w$ . We may identify the pairing  $\langle w, v \rangle_{(H^{1/2}, H^{-1/2})}$  as the integral  $\int_{B_1 \setminus \Gamma} \nabla w \cdot \eta = \int_{\Gamma} w \eta \cdot \nu$  where  $\nu$  is the normal of  $\Gamma$ . We will also use  $\text{sgn}(x, y)$  to mean  $y/|y|$  if  $x \geq 0$  and  $\text{sgn}(x, y) = 1$  if  $x < 0$ .

## 2 Preliminary Estimates.

In this section we gather various preliminary estimates about minimizers of the Mumford-Shah problem. We begin with a quantitative version of the result that the constant in to blow-up limit (8) is  $\sqrt{\frac{2}{\pi}}$ .

**Lemma 2.1.** *There exists a constant  $\epsilon_0 > 0$  such that if  $|\epsilon| \leq \epsilon_0$  then there exists a constant  $C_0$  and a pair  $(v, \Gamma_v)$  such that*

$$v(x) = \sqrt{\frac{2}{\pi}}(1 + \epsilon)r^{1/2} \sin(\phi/2) \text{ on } \partial B_1(0) \quad (10)$$

and

$$J(v, \Gamma_v) \leq J\left(\sqrt{\frac{2}{\pi}}(1 + \epsilon)r^{1/2} \sin(\phi/2), \Gamma_0\right) - C_0\epsilon^2.$$

*Remark:* Notice that the Lemma states that a minimizer in  $B_1(0)$  with boundary conditions of the form (10) can only be a homogeneous extension of the boundary data if  $\epsilon = 0$ . In particular, if  $(u, \Gamma_u)$  is  $\epsilon$ -close to a crack-tip with  $\epsilon = 0$  then

$$v(x) = \sqrt{\frac{2}{\pi}}r^{1/2} \sin(\phi/2).$$

*Proof of Lemma 2.1:* In this proof we let  $\mu = c\epsilon$  where  $c$  is a small constant. We will also use the notation  $O(\mu^2)$  for some function whose absolute value is smaller than  $C_1\mu^2$  for some fixed constant  $C_1$  - that is  $O(\mu^2)$  will be a uniform “big Oh notation”. From the proof we will see that  $C_1$  will only depend on the derivatives of  $r^{1/2} \sin(\phi/2)$  and is thus universal. We will assume that  $\epsilon > 0$  for definiteness. The proof consists of showing that, when  $\epsilon > 0$ , there exists a competitor for minimality with strictly larger (under inclusion) free discontinuity set if  $\epsilon > 0$ . If  $\epsilon < 0$  then an analogous argument works where one shortens the length of the crack instead.

We consider the ball  $\hat{B} = B_{1-\mu}(\mu e_1) \subset B_1(0)$  and use the notation

$$p(r, \phi) = \sqrt{\frac{2}{\pi}}(1 + \epsilon)r^{1/2} \sin(\phi).$$

Then

$$\begin{aligned} \int_{\hat{B}} |\nabla p|^2 + \mathcal{H}(\Gamma_0 \cap \hat{B}) &= \int_{B_{1-\mu}(0)} |\nabla p|^2 + \mathcal{H}(\Gamma_0 \cap B_{1-\mu}(0)) - \mu + O(\mu^2) = \\ &= (1 + \epsilon)^2(1 - \mu) + (1 - 2\mu) + O(\mu^2). \end{aligned} \quad (11)$$

To derive (11) we calculate the derivative

$$\frac{\partial}{\partial t} \int_{B_{1-\mu}(t)} |\nabla p|^2 \Big|_{t=0} = \int_{B_{1-\mu}(t)} \cos(\phi) |\nabla p|^2 = 0.$$

This gives that

$$\int_{\hat{B}} |\nabla p|^2 = \int_{B_{1-\mu}(0)} |\nabla p|^2 + O(\mu^2). \quad (12)$$

The calculation in (11) is just (12) together with an explicit calculation  $\mathcal{H}^1(\Gamma_0 \cap \hat{B}) = 1 - \mu$ .

Next we observe that at  $(r \cos(\phi) + \mu, r \sin(\phi)) \in \partial \hat{B}$

$$\begin{aligned} p(r \cos(\phi) + \mu, r \sin(\phi)) &= p(r, \phi) + \frac{\partial p(r, \phi)}{\partial x_1} \mu + O(\mu^2) = \\ &= \sqrt{\frac{2}{\pi}} \left(1 + \epsilon - \frac{\mu}{2}\right) r^{1/2} \sin(\phi/2) + O(\mu^2), \end{aligned}$$

where we used a Taylor expansion again.

Letting  $\hat{r}$  and  $\hat{\phi}$  be polar coordinates centered at the origin of  $\hat{B}$  we may define the function in  $\hat{B}$

$$w(\hat{r}, \hat{\phi}) = \sqrt{\frac{2}{\pi}} \left(1 + \epsilon - \frac{\mu}{2}\right) \hat{r}^{1/2} \sin(\hat{\phi}/2) + O(\mu^2)$$

such that  $w = p$  on  $\partial \hat{B}$ . Moreover we choose  $\Gamma_w = \{(\hat{r}, \pi); \hat{r} > 0\}$ . Then

$$\int_{\hat{B} \setminus \Gamma_w} |\nabla w|^2 + \mathcal{H}^1(\Gamma_w) = \left(1 + \epsilon - \frac{\mu}{2}\right)^2 (1 - \mu) + (1 - \mu) + O(\mu^2).$$

We can conclude that

$$\begin{aligned} &\int_{\hat{B}} |\nabla p|^2 + \mathcal{H}(\Gamma_0 \cap \hat{B}) - \left( \int_{\hat{B} \setminus \Gamma_w} |\nabla w|^2 + \mathcal{H}^1(\Gamma_w) \right) = \\ &= (1 + \epsilon)^2 (1 - \mu) + (1 - 2\mu) - \left(1 + \epsilon - \frac{\mu}{2}\right)^2 (1 - \mu) - (1 - \mu) + O(\mu^2) = \quad (13) \\ &= \mu\epsilon - \frac{5\mu^2}{4} + O(\mu^2) > 0 \end{aligned}$$

if  $0 < \mu < c\epsilon$  and  $\epsilon$  is small enough and  $c$  is a small constant. The calculation (13) shows that the function

$$v(x) = \begin{cases} p(x) & \text{in } B_1(0) \setminus \hat{B} \\ w(x) & \text{in } \hat{B} \end{cases}$$

satisfies the conclusion of the lemma.  $\square$

A simple perturbation result shows that any minimizer, not just minimizers with the limited boundary conditions specified by (10), satisfies a similar estimate. The formulation of this corollary differs somewhat of the formulation of the preceding Lemma - but the proofs are similar.

**Corollary 2.1.** *Let  $(u, \Gamma_u)$  be a minimizer to the Mumford-Shah problem and assume that*

$$\int_{B_1 \setminus \Gamma_u} \left| \nabla \left( u - \lambda \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) \right) \right|^2 \leq \epsilon^2 \quad (14)$$

for some small  $\epsilon > 0$ .

Then there exists a constant  $C_\Pi$  such that

$$1 - C_\Pi \sqrt{\lambda \epsilon} \leq |\lambda| \leq 1 + C_\Pi \sqrt{\lambda \epsilon}. \quad (15)$$

*Proof:* It is enough to show this for small  $\epsilon$ .

If  $u$  satisfies (14) then, by the trace theorem,

$$\left\| u|_{\partial B_1} - \lambda \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) \right\|_{H^{1/2}(\partial B_1(0) \setminus \Gamma_u)} \leq C\epsilon,$$

let us denote

$$w = u|_{\partial B_1} - \lambda \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) \text{ on } \partial B_1(0).$$

We know that there exists a function  $v$  such that

$$v = \lambda \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) \quad \text{on } \partial B_1(0)$$

and

$$J(v, \Gamma_v) \leq J \left( \lambda \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2), \Gamma_0 \right) - C_0(\lambda - 1)^2.$$

If we extend  $w$  to a harmonic function solving

$$\begin{aligned} \Delta w &= 0 & \text{in } B_1(0) \setminus \Gamma_v \\ \frac{\partial w}{\partial \nu} &= 0 & \text{on } \Gamma_v. \end{aligned}$$

Then

$$\begin{aligned} J(v + w, \Gamma_0) &\leq J(v, \Gamma_v) + C\epsilon^2 + 2 \int_{B_1(0) \setminus \Gamma_v} \nabla v \cdot \nabla w \leq \\ &\leq J \left( \lambda \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2), \Gamma_0 \right) - C_0(\lambda - 1)^2 + C_1 \lambda \epsilon. \end{aligned}$$

We may also use (14) to calculate

$$J(u, \Gamma_u) \geq J \left( \lambda \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2), \Gamma_0 \right) - C_2 \lambda \epsilon,$$

where we also used that  $\mathcal{H}^1(\Gamma_0) \leq \mathcal{H}^1(\Gamma_u)$ .

Since  $J(u, \Gamma_u) \leq J(v + w, \Gamma_0)$  we can conclude that

$$C_0(\lambda - 1)^2 \leq (C_1 + C_2) \lambda \epsilon,$$

which implies the Corollary.  $\square$

**Definition 2.1.** Let  $(u, \Gamma)$  be a minimizer to the Mumford-Shah problem in  $B_s$ , then we define the projection operator  $\Pi(u, \Gamma, s) = \Pi(u, s)$  to be the minimizer of the following expression

$$\min_{p \in \mathcal{P}(\Gamma)} \int_{B_s \setminus \Gamma} |\nabla(u - p)|^2$$

where

$$\mathcal{P}(\Gamma) = \left\{ p; p(r, \phi) = \sqrt{\frac{2}{\pi}} \lambda r^{\frac{1}{2}} \sin\left(\frac{\phi}{2} + \phi_0\right); \phi_0 \in (-\pi, \pi], \lambda \in \mathbb{R} \right\}.$$

We think of  $\Pi$  as a projection operator since

$$\int_{B_1(0) \setminus \Gamma_u} (\nabla(u - \Pi(u, 1))) \cdot \nabla v = 0$$

for any  $v \in \mathcal{P}$ . In particular we may conclude that

$$\int_{B_1(0) \setminus \Gamma_u} (\nabla(u - \Pi(u, 1))) \cdot \nabla(r^{1/2} \sin(\phi/2)) = 0$$

and similarly

$$\int_{B_1(0) \setminus \Gamma_u} (\nabla(u - \Pi(u, 1))) \cdot \nabla(r^{1/2} \cos(\phi/2)) = 0.$$

Next we formulate a lemma to control the Hausdorff measure of the free discontinuity set.

**Lemma 2.2.** Assume that  $(u, \Gamma)$  is  $\epsilon$ -close to a crack-tip. Then there is a constant  $C_{\mathcal{H}}$  such that

$$\mathcal{H}^1(\Gamma \cap B_1) \leq 1 + C_{\mathcal{H}}\epsilon. \quad (16)$$

*Proof:* The proof is very simple. In particular, we may compare the energy of  $(u, \Gamma)$  in  $B_1(0)$  to the competing pair  $(w, \Gamma_0)$  where  $\Gamma_0 = \{(x_1, 0); x_1 \in (-1, 0)\}$  and  $w$  is harmonic in  $B_1(0) \setminus \Gamma_0$ , has zero Neumann boundary data on  $\Gamma_0$  and equals  $u$  on the boundary  $\partial B_1(0)$ . From minimality we can conclude that

$$\begin{aligned} \int_{B_1(0) \setminus \Gamma_u} |\nabla u|^2 + \mathcal{H}^1(\Gamma_u \cap B_1) &\leq \int_{B_1 \setminus \Gamma_w} |\nabla w|^2 + \mathcal{H}^1(\Gamma_w \cap B_1) \leq \\ &\leq \int_{B_1 \setminus \Gamma_0} \left| \nabla \left( \lambda \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) \right) \right|^2 + C\epsilon + 1, \end{aligned} \quad (17)$$

where we used  $\mathcal{H}^1(\Gamma_0) = 1$  in the last estimate. Since  $(u, \Gamma)$  is  $\epsilon$ -close to a crack-tip it follows that

$$\begin{aligned} J(u, \Gamma_u) &= \int_{B_1 \setminus \Gamma_u} \left| \nabla \left( u - \lambda \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) \right) \right|^2 - \\ &\quad - 2 \int_{B_1 \setminus \Gamma_u} \nabla \left( u - \lambda \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) \right) \cdot \nabla u + \end{aligned}$$

$$\begin{aligned}
& + \int_{B_1 \setminus \Gamma_u} \left| \nabla \left( \lambda \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) \right) \right|^2 + \mathcal{H}^1(\Gamma_u) \geq \\
& \geq \int_{B_1 \setminus \Gamma_u} \left| \nabla \left( \lambda \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) \right) \right|^2 - C\epsilon + \mathcal{H}^1(\Gamma_u). \tag{18}
\end{aligned}$$

The estimate (17) and the estimate ending in (18) clearly implies that

$$\mathcal{H}^1(\Gamma_u \cap B_\mu) \leq \mathcal{H}^1(\Gamma_0 \cap B_1) + C\epsilon.$$

□

**Corollary 2.2.** *Assume that  $(u, \Gamma)$  is  $\epsilon$ -close to a crack-tip. Then there is a line,  $l$ , from the origin to  $\partial B_1(0)$  such that*

$$\sup_{x \in \Gamma_u} \text{dist}(x, l) \leq C\sqrt{\epsilon}.$$

*Proof:* We may rotate the coordinate system so that  $(-1, 0) \in \Gamma_u$ . Let  $(x_1, x_2) \in \Gamma_u$  then

$$\sqrt{(1+x_1)^2 + x_2^2} + \sqrt{x_1^2 + x_2^2} \leq \mathcal{H}^1(\Gamma \cap B_1) \leq 1 + C\epsilon. \tag{19}$$

The left side in (19) attains its minimal value for fixed  $x_2$  when  $x_1 = -1/2$ . Thus

$$\sqrt{1 + 4x_2^2} \leq \mathcal{H}^1(\Gamma \cap B_1) \leq 1 + C\epsilon.$$

This implies that  $|x_2| \leq C\sqrt{\epsilon}$ . Similarly, for fixed  $x_1$  the minimal value of (19) is attained for  $x_2 = 0$  which implies that  $x_1 < C\sqrt{\epsilon}$ . This implies the corollary. □

**Lemma 2.3.** *For each  $\tau > 0$  there exists an  $\epsilon_\tau > 0$  such that if  $(u, \Gamma)$  is  $\epsilon$ -close to a crack-tip for some  $\epsilon < \epsilon_\tau$  then  $\Gamma$  is a graph in  $B_1 \setminus B_\tau$ . That is*

$$\Gamma \cap B_1 = \{(x_1, f(x_1)); x_1 \in (-1, -\tau)\} \text{ in the set } -1 < x_1 \leq -\tau.$$

*Furthermore*

$$|f'(x_1)| \leq \sigma(\epsilon) \text{ for } x_1 \leq -\tau$$

*for some modulus of continuity  $\sigma$ .*

*Proof:* The proof is a direct application of Theorem 1.1. We let  $\delta > 0$  be a small constant, in particular we assume that  $0 < \delta < \gamma_0$ , where  $\gamma_0$  is as in Theorem 1.1. We also consider a small ball  $B_s(-te_1)$  with  $0 < s < t$  and  $s$  small enough so that

$$\frac{1}{s} \int_{B_s(-te_1) \setminus \Gamma_0} \left| \nabla \left( \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) \right) \right|^2 \leq \frac{\delta}{2}. \tag{20}$$

Notice that we may choose  $s = c\delta t$  for some fixed small constant  $c$ .

Clearly if  $\epsilon$  is small enough, depending only on  $s$  and  $\delta$ , then

$$\frac{1}{s} \int_{B_s(-te_1) \setminus \Gamma_u} |\nabla u|^2 \leq \delta < \gamma_0. \tag{21}$$

Furthermore by Corollary 2.2, if  $\epsilon$  is small enough,

$$\Gamma_u \subset \{(x_1, x_2); |x_2| < s^2\delta\}. \quad (22)$$

We may conclude that for any  $0 < \delta < \gamma_0$  then if  $\epsilon$  is small enough so that (22) and (21) are satisfied then it follows, from Theorem 1.1, that  $\Gamma_u$  is a  $C^{1,\alpha}$ -graph in  $B_{s/2}(-te_1)$  and

$$|f'(-t)| \leq Cs^{1/4} \left(1 + \frac{\delta}{s}\right)^{1/2} = Ct^{1/4}\delta^{1/4} \sqrt{1 + \frac{1}{ct}} \leq C\sqrt{1 + \frac{1}{c\tau}}\delta^{1/4} \quad (23)$$

on the set  $-t \leq \tau < 0$ .

But  $\delta$  is any constant s.t.  $0 < \delta < \gamma_0$  and therefore (23) implies that  $|f'(-t)|$  can be made arbitrarily small on the set  $\{-t \leq -\tau\}$  if  $\epsilon$  is small enough.

This proves the Lemma.  $\square$

The following monotonicity formula was first proved in [8].

**Lemma 2.4.** *Let  $(u, \Gamma_u)$  be a minimizer of the Mumford-Shah energy and assume that  $\Gamma_u$  has finitely many connected components. Then*

$$\frac{1}{r} \int_{B_r(0) \setminus \Gamma_u} |\nabla u|^2 \quad (24)$$

*is non-decreasing in  $r$ . Furthermore the functional in (24) is constant only if  $u(x) - u(0)$  is  $r^{1/2}$ -homogeneous.*

*In particular, if  $(u, \Gamma_u)$  is  $\epsilon$ -close to a crack-tip then (24) is non decreasing.*

*Proof:* For the proof of the first part of the lemma see [8]. The second part follows since, by assumption, a minimizer that is  $\epsilon$ -close to a crack-tip is a minimizer and  $\Gamma_u \cap B_1(0)$  has one component.  $\square$

### 3 Linearization.

From now on we will assume that we have a sequence of minimizers  $(u^j, \Gamma_j)$  that are  $\epsilon_j$  close to a crack-tip,  $\epsilon_j \rightarrow 0$ . We will write

$$u^j = \Pi(u^j, 1) + \epsilon_j v^j,$$

by definition we have that  $\|\nabla v^j\|_{L^2(B_1 \setminus \Gamma_j)} = 1$ . We remark that

$$\Gamma_j \setminus B_{\tau_j} = \{(x, \epsilon_j f_j(x)); x \in (-1, -\tau_j)\} \cap B_1 \quad (25)$$

where  $\tau_j \rightarrow 0$  and  $f_j \in C^1$ . The statement (25) follows from Lemma 2.3.

Our aim is to derive a system of equations for  $v^0 = \lim_{j \rightarrow \infty} v^j$  and  $f_0 = \lim_{j \rightarrow \infty} f_j$ . That  $v^j \rightarrow v^0$ , at least for a subsequence, is clear by the weak compactness of  $W^{1,2}$ . We will need to show that the convergence is strong. We will prove strong convergence in Section 5. In this section we will only derive the equations the limit functions solve and show that the convergence  $v^j \rightarrow v^0$  is strong away from the crack-tip, that is in the set  $B_b(0) \setminus B_a(0)$  for some  $1 > b > a > 0$ .

In order to handle the problems with  $\Gamma_{u^j}$  being different for each  $j$  which means that each  $u^j \in W^{1,2}(B_1 \setminus \Gamma_{u^j})$  belongs to a different Sobolev space we need to consider the regular part of the gradient as an  $L^2$  function.

**Definition 3.1.** We will use the notation  $\text{Reg}(\nabla v^j)$  and  $\text{Sing}(\nabla v^j)$  to denote the regular part of the measure  $\nabla v^j$  when  $v^j$  is considered to be a function of bounded variation in  $B_1$ . In particular,  $\text{Reg}(\nabla v^j)$  will be an  $L^2$ -function agreeing with  $\nabla v^j$  on  $B_1 \setminus \Gamma_j$ .

**Proposition 3.1.** [LINEARIZATION AT THE CRACK-TIP.] Let  $(u^j, \Gamma_j)$  be a sequence of minimizers to the Mumford-Shah problem that are  $\epsilon_j$ -close to a crack tip for some sequence  $\epsilon_j \rightarrow 0$ . Furthermore, let

$$v^j(x) = \frac{u^j(x) - \Pi(u^j, 1)}{\epsilon_j}. \quad (26)$$

Then there exists a subsequence, which we still denote  $v^j$ , such that  $v^j \rightarrow v^0$  strongly in  $L^2$  and  $\text{Reg}(\nabla v^j) \rightarrow \text{Reg}(\nabla v^0)$  strongly in  $L^2(B_b(0) \setminus B_a(0))$  for any  $0 < a < b < 1$ . If  $f_j$  is given by (see (25))

$$\Gamma_j \setminus \{|x_1| < \tau_j\} = \{(x_1, \epsilon_j f_j(x_1)); x_1 \leq -\tau_j\}$$

then  $f_j \rightarrow f_0$  for some function  $f_0 \in C([-1, 0]) \cap W^{1,2}((-b, -a))$  for any  $0 < a < b < 1$ .

Furthermore  $v^0$  and  $f_0$  satisfies the following equations

$$\begin{aligned} \Delta v^0 &= 0 && \text{in } B_1 \setminus \{x_1 < 0, x_2 = 0\} \\ -\frac{\partial v^0(x_1, 0^+)}{\partial x_2} &= \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial x_1} \left( \frac{1}{\sqrt{-x_1}} f_0(x_1) \right) && \text{for } x_1 < 0 \\ -\frac{\partial v^0(x_1, 0^-)}{\partial x_2} &= -\frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial x_1} \left( \frac{1}{\sqrt{-x_1}} f_0(x_1) \right) && \text{for } x_1 < 0 \\ \frac{\partial^2 f(x_1)}{\partial x_1^2} &= -\sqrt{\frac{2}{\pi}} \frac{1}{r} \left( \frac{\partial v^0(x_1, 0^+)}{\partial x_1} + \frac{\partial v^0(x_1, 0^-)}{\partial x_1} \right) && \text{for } x_1 < 0. \end{aligned} \quad (27)$$

*Proof:* We will split the proof into several claims. We begin with proving that the limit functions satisfy the equations.

**Claim 1:** Assume that  $\text{Reg}(\nabla v^j) \rightarrow \text{Reg}(\nabla v^0)$  in  $W^{1,2}(B_b(0) \setminus B_a(0))$  and  $f_j \rightarrow f_0$  weakly in  $W^{1,2}((-b, -a))$ . Then the pair  $(v^0, f^0)$  satisfy the equations (27) specified in the Proposition.

*Proof of Claim 1:* We start by doing a domain variation (see (3)) of the Mumford-Shah energy, with  $\eta(x) = \psi(x)e_2$  with  $\psi \in C_c^\infty(B_1(0) \setminus B_{\mu_0}(0))$  and  $D_2\psi(x) = 0$  close to  $\Gamma_{u^j}$ , and derive that

$$\begin{aligned} 0 &= \int_{B_1(0) \setminus \Gamma_{u^j}} \left( |\nabla(\Pi(u^j) + \epsilon_j v^j(x))|^2 \frac{\partial \psi}{\partial x_2} - \right. \\ &\quad \left. - 2(\nabla(\Pi(u^j) + \epsilon_j v^j) \cdot e_2)(\nabla(\Pi(u^j) + \epsilon_j v^j) \cdot \nabla \psi) \right) + \\ &\quad + \int_0^1 \frac{\epsilon_j f_j'(x_1)}{\sqrt{1 + \epsilon_j^2 |f_j'(x_1)|^2}} \frac{\partial \psi(x)}{\partial x_1} = \\ &= \left[ \int_{B_1(0) \setminus \Gamma_{u^j}} \left( |\nabla \Pi|^2 \frac{\partial \psi}{\partial x_2} - 2(\nabla \Pi \cdot e_2)(\nabla \Pi \cdot \nabla \psi) \right) \right] + \end{aligned} \quad (28)$$



$$+ \epsilon_j \int_{B_1 \setminus \Gamma_{u^j}} (2\nabla \Pi \cdot \nabla v^j - 2(\nabla \Pi \cdot e_2)(\nabla v^j \cdot \nabla \psi) - 2(\nabla v^j \cdot e_2)(\nabla \Pi \cdot \nabla \psi)) + \quad (29)$$

$$+ \epsilon_j \int_0^1 \frac{f'_j(x_1)}{\sqrt{1 + \epsilon_j^2 |f'_j(x_1)|^2}} \frac{\partial \psi(x)}{\partial x_1} + \quad (30)$$

$$+ \epsilon_j^2 \int_{B_1(0) \setminus \Gamma_{u^j}} (|\nabla v^j|^2 - 2(\nabla v^j \cdot e_2)(\nabla v^j \cdot \psi)) . \quad (31)$$

If we make an integration by parts in (28) we arrive at

$$\int_{\Gamma_u^\pm} |\nabla \Pi|^2 \psi(\nu^\pm \cdot e_2) - 2(\nabla \Pi \cdot e_2)(\nabla \Pi \cdot \nu^\pm) \psi. \quad (32)$$

Notice that

$$\nabla \Pi(u^j) = \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{1}{r^{1/2}} (-\sin(\phi/2), \cos(\phi/2)). \quad (33)$$

In particular, since the value of  $\phi$  differs by  $2\pi$  on  $\Gamma^\pm$ , we can conclude that  $\nabla \Pi(u^j)|_{\Gamma^+} = -\nabla \Pi(u^j)|_{\Gamma^-}$ . Since also  $\nu^+ = -\nu^-$  it follows that the value of (32) is identically zero.

Moreover, the term in (31) is of order  $\epsilon_j^2$  since  $\|\text{Reg}(\nabla v^j)\|_{L^2} = 1$ . This means that the terms in (29)-(30) must tend to zero as  $j \rightarrow \infty$ . We can thus conclude that

$$\int_{B_1 \setminus \Gamma_{u^j}} (2\nabla \Pi \cdot \nabla v^j - 2(\nabla \Pi \cdot e_2)(\nabla v^j \cdot \nabla \psi) - 2(\nabla v^j \cdot e_2)(\nabla \Pi \cdot \nabla \psi)) + \quad (34)$$

$$+ \int_0^1 \frac{f'_j(x_1)}{\sqrt{1 + \epsilon_j^2 |f'_j(x_1)|^2}} \frac{\partial \psi(x)}{\partial x_1} = \quad (35)$$

$$= \int_{\Gamma_j^\pm} [2(\nabla \Pi \cdot \nabla v^j)(\nu^\pm \cdot e_2) - 2(\nabla \Pi \cdot e_2)(\nabla v^j \cdot \nu^\pm) - 2(\nabla v^j \cdot e_2)(\nabla \Pi \cdot \nu^\pm)] \psi +$$

$$+ \int_0^1 \frac{f'_j(x_1)}{\sqrt{1 + \epsilon_j^2 |f'_j(x_1)|^2}} \frac{\partial \psi(x)}{\partial x_1} = o(1) \quad (36)$$

Using the expression (33) and that  $\sin(\phi/2) = \pm 1 + O(\sqrt{\epsilon_j})$  and  $\cos(\phi) = O(\sqrt{\epsilon_j})$  on  $\Gamma_u^\pm \cap B_b \setminus B_a$  since  $|f_j| \leq C\sqrt{\epsilon_j}$  by Corollary 2.2. We can conclude that

$$o(1) = \int_0^1 \left( \sqrt{\frac{2}{\pi}} \frac{1}{r} \left( \frac{\partial v^j}{\partial x_1} \Big|_{\Gamma_u^+} + \frac{\partial v^j}{\partial x_1} \Big|_{\Gamma_u^-} \right) \psi + f'_j \frac{\partial \psi(x)}{\partial x_1} \right). \quad (37)$$

From (36), (37) and

$$\sqrt{1 + \epsilon_j^2 |f'_j|^2} = 1 + o(1). \quad (38)$$

we conclude that, in the weak sense,

$$f''_0(x_1) = -\sqrt{\frac{2}{\pi}} \frac{1}{r} \left( \frac{\partial v^0(x_1, 0^+)}{\partial x_1} + \frac{\partial v^0(x_1, 0^-)}{\partial x_1} \right).$$

This is the first of the limit equations for  $v^0$  and  $f_0$ .

To derive the second equation for  $v^0$  and  $f_0$  we use that on  $\Gamma_{u^j}^\pm$

$$0 = \nabla u^j \cdot \nu = \nabla (\Pi(u^j) + \epsilon_j v^j) \cdot (\epsilon_j f'_j, -1). \quad (39)$$

That is

$$\epsilon_j \frac{\partial v^j}{\partial x_1} f'_j - \frac{\partial v^j}{\partial x_2} = \frac{1}{\epsilon_j} (\nabla \Pi \cdot (-\epsilon_j f'_j, 1)). \quad (40)$$

If we use (33) in (40) we see that on  $\Gamma_{u^j}^+$ , where  $\phi = \pi - \frac{\epsilon_j f_j}{r} + O\left(\left(\frac{\epsilon_j f_j}{r}\right)^3\right)$ ,

$$\begin{aligned} & \frac{1}{\epsilon_j} (\nabla \Pi \cdot (-\epsilon_j f'_j, 1)) = \\ &= \frac{1}{2\epsilon_j} \sqrt{\frac{2}{\pi}} \frac{1}{r} \left( \epsilon_j \sin\left(\frac{\pi - \epsilon_j f_j/r}{2}\right) f'_j + \cos\left(\frac{\pi - \epsilon_j f_j/r}{2}\right) \right) + o(\epsilon_j) = \\ &= \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{1}{r} \left( f'_j(x_1) + \frac{f_j(x_1)}{2r} \right) + o(\epsilon_j). \end{aligned} \quad (41)$$

Equations (39), (40) and (41) together implies that, in the weak sense,

$$-\epsilon_j f'_j \frac{\partial v^j}{\partial x_1} + \frac{\partial v^j(x_1, 0^-)}{\partial x_2} = \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{1}{r} \left( f'_j(x_1) + \frac{f_j(x_1)}{2r} \right) + o(\epsilon_j) \quad (42)$$

Passing to the limit in (42) we may conclude that

$$-\frac{\partial v^0(x_1, 0^+)}{\partial x_2} = \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{1}{r} \left( f'_0(x_1) + \frac{f_0(x_1)}{2r} \right).$$

Similarly, on  $\Gamma_u^-$ , where  $\phi = -\pi - \frac{\epsilon_j f_j}{r} + o\left(\left(\frac{\epsilon_j f_j}{r}\right)^3\right)$ , we can conclude, after passing to the limit, that

$$-\frac{\partial v^0(x_1, 0^-)}{\partial x_2} = -\frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{1}{r} \left( f'_0(x_1) + \frac{f_0(x_1)}{2r} \right).$$

This proves claim 1.

**Claim 2:** The function  $f_j$  satisfies

$$\left\| \frac{\partial}{\partial x_1} \left( \frac{1}{\sqrt{-x_1}} f_j(x_1) \right) \right\|_{H^{-1/2}} \leq 1,$$

which implies that there exists a constant  $\gamma_j$  such that

$$\|f_j - \gamma_j \sqrt{-x_1}\|_{L^2} \leq C.$$

*Proof of Claim 2:* From the equality (42) we can conclude that, for any  $\zeta \in C_0^\infty(B_b \setminus B_a)$ ,

$$\int_{\Gamma_{u^j}} \left( \epsilon_j f'_j \frac{\partial v^j}{\partial x_1} - \frac{\partial v^j(x_1, 0^-)}{\partial x_2} \right) \zeta d\mathcal{H}^1|_{\Gamma_{u^j}} + \quad (43)$$

$$+ \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_{\Gamma_{u^j}} \frac{\partial}{\partial x_1} \left( \frac{1}{\sqrt{-x_1}} f(x_1) \right) \zeta d\mathcal{H}^1|_{\Gamma_{u^j}} = o(1). \quad (44)$$

Notice that, by an integration by parts using that of  $\Gamma_{u^j}$  is  $(\epsilon f'_j, 1)$  c.f. (39),

$$\begin{aligned} & \int_{\Gamma_{u^j}} \left( \epsilon_j f'_j \frac{\partial v^j}{\partial x_1} - \frac{\partial v^j(x_1, 0^-)}{\partial x_2} \right) \zeta = \\ & = \int_{B_1 \setminus \Gamma_u} \nabla v^j \cdot \nabla \zeta \leq \|\nabla v^j\|_{L^2} \|\nabla \zeta\|_{L^2} \leq \|\nabla \zeta\|_{L^2}. \end{aligned} \quad (45)$$

we can conclude from (43)-(44) and (45) that

$$\int_{\Gamma_{u^j}} \frac{\partial}{\partial x_1} \left( \frac{1}{\sqrt{-x_1}} f(x_1) \right) \zeta \leq \left( \int_{B_1 \setminus \Gamma} |\nabla \zeta|^2 \right)^{1/2} + o(1) \quad (46)$$

Choosing the  $\zeta$  that maximizes the left side in (46) under the constraint  $\|\nabla \zeta\|_{L^2} \leq 1$  we can conclude that

$$\left\| \frac{\partial}{\partial x_1} \left( \frac{1}{\sqrt{-x_1}} f_j(x_1) \right) \right\|_{H^{-1/2}} \leq 1.$$

It follows that  $\frac{1}{\sqrt{-x_1}} f_j(x_1) \in H^{1/2}$  modulo solutions,  $h(x_1)$ , to the ODE

$$\frac{\partial}{\partial x_1} \left( \frac{1}{\sqrt{-x_1}} h(x_1) \right) = 0.$$

**Claim 3.** *The following estimate holds:*

$$\left( \int_{-b}^{-a} \left( \frac{f'_j(x_1)}{\sqrt{1 + \epsilon_j^2 |f'_j|^2}} - \left( \frac{f'_j(x_1)}{\sqrt{1 + \epsilon_j^2 |f'_j|^2}} \right)_{(-b, -a)} \right)^2 \right)^{1/2} \leq C$$

where  $(h(x_1))_{(-b, -a)}$  indicates the average of the function  $h(x_1)$  on  $(-b, -a)$ .

*Proof of claim 3:* The proof follows the same lines as the proof of claim 2. From (34), (35) and (36) we deduce that

$$\int_{B_1 \setminus \Gamma_u} (2\nabla \Pi \cdot \nabla v^j - 2(\nabla \Pi \cdot e_2)(\nabla v^j \cdot \nabla \psi) - 2(\nabla v^j \cdot e_2)(\nabla \Pi \cdot \nabla \psi)) + \quad (47)$$

$$+ \int_0^1 \frac{f'_j(x_1)}{\sqrt{1 + \epsilon_j^2 |f'_j(x_1)|^2}} \frac{\partial \psi(x)}{\partial x_1} = o(1). \quad (48)$$

Choosing  $\xi$  such that  $\xi = \frac{\partial \psi}{\partial x_1}$  and  $\|\nabla \xi\|_{L^2} = 1$  and notice that this implies that  $\int \xi dx_1 = 0$  we see that

$$\left| \int_0^1 \frac{f'_j(x_1)}{\sqrt{1 + \epsilon_j^2 |f'_j(x_1)|^2}} \xi \right| \leq C$$

for any function  $\xi \in H^{1/2} \subset L^2$  with zero average. This implies the claim.

**Claim 4.** For  $j$  large enough the function  $f_j \in W^{1,2}(-b, -a)$  with the estimate  $\|f_j\|_{W^{1,2}(-b, -a)} \leq C_{-b, -a}$ .

*Proof of claim 4:* If we denote

$$\tau_j = \left( \frac{f'_j(x_1)}{\sqrt{1 + \epsilon_j^2 |f'_j|^2}} \right)_{(-b, -a)}$$

then, using that  $\sqrt{1 + |f'_j|^2} \approx 1$  in  $[-b, -a]$  when  $j$  is large enough (cf. Lemma 2.3), from claim 3 we can deduce that

$$\left( \int_{-b}^{-a} \left( f'_j(x_1) - \sqrt{1 + \epsilon_j^2 |f'_j|^2} \tau_j \right)^2 \right)^{1/2} \leq C$$

and from claim 2 that

$$\|f_j - \gamma_j \sqrt{-x_1}\|_{L^2} \leq C.$$

Clearly both  $\tau_j$  and  $\gamma_j$  must be bounded. This implies the claim.

**Claim 5:** The sequence  $\frac{\partial v^j}{\partial \nu_j^\pm}(x_1, \epsilon_j f_j(x_1))$ , where  $\nu_j^\pm$  are the upper and lower normals of  $\Gamma_j$ , are uniformly bounded in  $L^2(-b, -a)$ .

*Proof of Claim 5:* Since  $\Gamma_j$  is a  $C^{1,\alpha}$  graph for  $x_1 \in (-b, -a)$  and

$$0 = \frac{\partial u^j}{\partial \nu_j^\pm} = \frac{\partial \Pi}{\partial \nu_j^\pm} + \epsilon_j \frac{\partial v^j}{\partial \nu_j^\pm}$$

we can deduce that

$$\frac{\partial v^j}{\partial \nu_j^\pm} = -\frac{1}{\epsilon_j} \frac{\partial \Pi}{\partial \nu_j^\pm}.$$

Expressing  $r^{1/2} \sin(\phi/2)$  in Cartesian coordinates<sup>1</sup>,

$$\Pi(u^j, 1) = c_j \frac{1}{\sqrt{2}} \operatorname{sgn}(x_1, x_2) \sqrt{\sqrt{x_1^2 + x_2^2} - x_1} \quad (49)$$

for some constant  $c_j$ . Since  $\nabla \Pi$  is an  $L^2$ -projection of  $\nabla u^j$  it follows that  $\|\nabla \Pi(u^j, 1)\|_{L^2} \leq \|\nabla u^j\|_{L^2}$ . We may therefore conclude that the numbers  $c_j$  are uniformly bounded.

It follows, from (49) and the expression of the normal  $\nu_j^\pm$  in terms of  $f_j$ , that

$$\begin{aligned} \frac{\partial \Pi}{\partial \nu_j^\pm} &= \pm \frac{\nabla \Pi \cdot (-\epsilon_j f'_j, 1)}{\sqrt{1 + \epsilon_j^2 |f'_j|^2}} = \\ &= \frac{c_j \operatorname{sgn}(x_1, x_2)}{2\sqrt{2} \sqrt{\sqrt{x_1^2 + \epsilon_j^2 |f_j(x_1)|^2} - x_1}} \left[ \left( \frac{-x_1}{\sqrt{x_1^2 + \epsilon_j^2 |f_j|^2}} + 1 \right) f'_j + \frac{f_j}{\sqrt{x_1^2 \epsilon_j^2 |f_j|^2}} \right] \epsilon_j. \end{aligned}$$

<sup>1</sup>By  $\operatorname{sgn}(x_1, x_2)$  we mean  $x_2/|x_2|$  if  $x_1 \geq 0$  and  $\operatorname{sgn}(x_1, x_2) = 1$  if  $x_1 < 0$ .

In particular, a simple calculation shows that

$$\left| \frac{\partial v^j}{\partial \nu_j^\pm} \right| = \frac{1}{\epsilon_j} \left| \frac{\partial \Pi}{\partial \nu_j^\pm} \right| \leq c_j C_{(-b, -a)} (|f'_j| + |f_j|).$$

The claim follows from the boundedness of  $f_j$  in  $W^{1,2}(-b, -a)$  and the uniform bound on  $c_j$ .

**Claim 6:**  $\text{Reg}(\nabla v^j)$  converges strongly in  $L^2(B_b \setminus B_a)$ .

*Proof of claim 6:* We notice that since  $u^j$  has zero Neumann data on  $\Gamma_u^\pm$  it follows that

$$\nabla v^j \cdot \nu_j^\pm = -\frac{1}{\epsilon_j} \nabla \Pi(u^j) \cdot \nu_j^\pm \text{ on } \Gamma_u^\pm \cap B_{(1+b)/2}(0) \setminus B_{a/2}(0).$$

A direct calculation shows that

$$\nabla \Pi(u^j) \cdot \nu_j^\pm = \frac{1}{2\epsilon_j} \sqrt{\frac{2}{\pi}} \frac{1}{r} \left( \epsilon_j \sin \left( \frac{\pi - \epsilon_j f_j / r}{2} \right) f'_j + \cos \left( \frac{\pi - \epsilon_j f_j / r}{2\sqrt{1 + \epsilon_j^2 |f'_j|^2}} \right) \right).$$

It follows, from  $f_j \in W^{1,2}((-1+b)/2, -a/2)$  that

$$\nabla \Pi(u^j) \cdot \nu_j^\pm \in L^2(\Gamma_u^\pm \cap B_{(1+b)/2}(0) \setminus B_{a/2}(0)).$$

In particular, the non-tangential maximal function of  $\nabla v^j$  is an  $L^2$ -function. From this it easily follows that

$$\|\nabla v^j\|_{L^2(\{|x_2| \leq \delta\} \cap B_b(0) \setminus B_a(0))} \leq \sigma(\delta),$$

if  $j$  is large enough and  $\sigma(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ .

By the triangle inequality we can conclude that

$$\begin{aligned} & \lim_{j \rightarrow \infty} \|\nabla (\text{Reg}(v^j) - \text{Reg}(v^0))\|_{L^2(B_b \setminus B_a)} \leq \\ & \leq \lim_{j \rightarrow \infty} \left( \|\nabla (\text{Reg}(v^j) - \text{Reg}(v^0))\|_{L^2(B_b \setminus (B_a \cup \{|x_2| \leq \delta\}))} + \right. \\ & \quad \left. + \|\nabla (\text{Reg}(v^j) - \text{Reg}(v^0))\|_{L^2(B_b \cap \{|x_2| \leq \delta\} \setminus B_a)} \right) \leq \sigma(\delta) \end{aligned}$$

since  $v^j$  converges uniformly in  $C^1$  in the compact set  $B_b \setminus (B_a \cup \{|x_2| \leq \delta\})$ . Since  $\delta > 0$  is arbitrary we can conclude that

$$\lim_{j \rightarrow \infty} \|\nabla (\text{Reg}(v^j) - \text{Reg}(v^0))\|_{L^2(B_b \setminus B_a)} = 0.$$

This proves the claim.  $\square$

## 4 Analysis of the Linearized System.

In order to derive any information from Proposition 3.1 we need to understand the linear system that  $(v^0, f_0)$  solves. The aim of this section is to prove the following Proposition and a simple corollary stated at the end of the section.

**Proposition 4.1.** *For each  $g \in H^{1/2}(\partial B_1(0) \setminus \{(-1, 0)\})$  and  $t \in \mathbb{R}$  there exists a weak solution  $(v(x_1, x_2), f(x_1)) \in W^{1,2}(B_1(0) \setminus \{(x_1, 0); x_1 \leq 0\}) \times W^{1,2}((-1, 0))$  to the following boundary value problem*

$$\begin{aligned} \Delta v &= 0 && \text{in } B_1 \setminus \{x_1 < 0, x_2 = 0\} \\ -\frac{\partial v(x_1, 0^+)}{\partial x_2} &= \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial x_1} \left( \frac{1}{\sqrt{-x_1}} f(x_1) \right) && \text{for } x_1 < 0 \\ -\frac{\partial v(x_1, 0^-)}{\partial x_2} &= -\frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial x_1} \left( \frac{1}{\sqrt{-x_1}} f(x_1) \right) && \text{for } x_1 < 0 \\ \frac{\partial^2 f(x_1)}{\partial x_1^2} &= -\sqrt{\frac{2}{\pi}} \frac{1}{r} \left( \frac{\partial v(x_1, 0^+)}{\partial x_1} + \frac{\partial v(x_1, 0^-)}{\partial x_1} \right) && \text{for } x_1 < 0 \\ v &= g && \text{on } \partial B_1(0) \setminus \{(-1, 0)\} \text{ and} \\ f(-1) &= t. \end{aligned} \tag{50}$$

Furthermore, if  $(v, f)$  satisfies the following estimates, for any constants  $C_v$  and  $C_f$ ,

$$\frac{1}{r} \int_{B_r(0) \setminus \{(x_1, 0); x_1 \leq 0\}} |\nabla v|^2 \leq C_v \ln(1/r), \tag{51}$$

$$|f(x_1)| \leq C_f |x_1| \ln(|x_1|) \quad \text{for } x_1 < 0 \tag{52}$$

and

$$|f'(x_1)| \leq C_f |\ln(|x_1|)| \quad \text{for } x_1 < 0 \tag{53}$$

then the solution is unique and we may express  $v$  and  $f$ ,  $v$  in polar coordinates,

$$v(r, \phi) = a + a_0 r^{1/2} \cos(\phi/2) + \sum_{k=1}^{\infty} a_k r^{\alpha_k} \cos(\alpha_k \phi) + \sum_{k=1}^{\infty} b_k r^{k-1/2} \sin((k-1/2)\phi) \tag{54}$$

and

$$f(x_1) = a_0 2 \sqrt{\frac{\pi}{2}} x_1 + \sum_{k=1}^{\infty} a_k 2 \sqrt{\frac{\pi}{2}} \sin(\alpha_k \pi) |x_1|^{\alpha_k + \frac{1}{2}} \tag{55}$$

for some constants  $a_k$  and  $b_k$ . Here  $\alpha_k \geq 0$  are the positive solutions,  $\alpha_k \in (k, k+1/2)$ , to the following equation

$$\tan(\alpha\pi) = \frac{2}{\pi} \frac{\alpha}{\alpha^2 - \frac{1}{4}}. \tag{56}$$

**Remark:** The values of  $C_v$  and  $C_f$  are not important in this Proposition - as long as the solution  $v$  does not blow up to fast at the origin and the derivatives of  $f$  does not blow up to fast the solution is unique.

Again the proof is quite long and we will therefore split it into several Lemmata. Our goal is to show that the limit  $v^0$  in Proposition 3.1 can be expressed as a series of homogeneous functions as in (54). To that end we begin to derive an expression of all homogeneous solutions to (50) in the next sub-section. In sub-section 4.2 we will show that these homogeneous functions span  $L^2(\partial B_1(0) \setminus (-1, 0))$  and thus  $H^{1/2}(\partial B_1(0) \setminus (-1, 0))$ . It follows that, for any boundary data  $g$ , we can find a solution  $(u, f)$  such that  $u = g$  on  $\partial B_1(0)$ . However, we also need to specify the boundary data of  $f(x_1)$  at  $x_1 = -1$ . We show that that is possible in sub-section 4.4. This shows that we may find a solution for each boundary data  $g$  on  $\partial B_1(0)$  and  $t = f(-1) \in \mathbb{R}$ . We also need to show the uniqueness of these solutions in order to conclude that the particular solution  $(u^0, f_0)$ , the one we get from the linearization in Proposition 3.1, has the desired form. We show uniqueness in sub-section 4.3. The proof of proposition 4.1 is then a simple consequence of the sub-sections 4.1-4.4.

### 4.1 Homogeneous solutions to (50).

**Lemma 4.1.** *Let  $(v, f)$  be a homogeneous solution to (50) then*

$$(v, f) = \left( ar^{\alpha_k} \cos(\alpha_k \phi), 2a\sqrt{\frac{\pi}{2}} \sin(\alpha_k \pi) r^{\alpha_k + \frac{1}{2}} \right)$$

where  $\alpha_k$  is a solution to

$$\tan(\alpha\pi) = \frac{2}{\pi} \frac{\alpha}{\alpha^2 - \frac{1}{4}}$$

or

$$v(r, \phi) = ar^{1/2} \cos(\phi/2) \text{ and } f(x_1) = 2\sqrt{\frac{\pi}{2}} ax_1$$

or

$$v = br^{k-1/2} \sin((k-1/2)\phi) \text{ and } f = 0$$

where  $k \in \mathbb{N}$ .

*Proof:* We aim to derive expressions for the homogeneous solutions to the system (50). Let us therefore assume that  $v(r, \phi) = r^\alpha \Phi(\phi)$  and  $f(r) = cr^\beta$  for some constants  $\alpha, \beta$  and  $c$ . Since  $v$  is harmonic it follows that  $\Phi(\phi) = a \cos(\alpha\phi) + b \sin(\alpha\phi)$  for some constants  $a$  and  $b$ .

If we first consider part of the homogeneous solution that is even in  $x_2$ . We see that the equations involving  $f$  in (50) reduces to

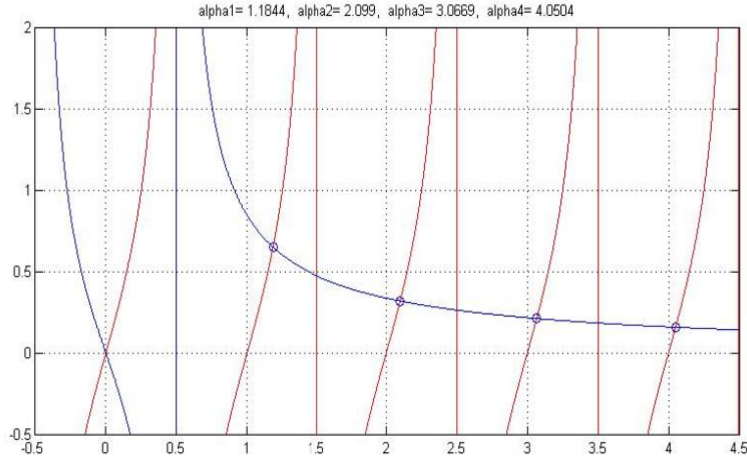
$$\begin{aligned} \alpha c r^{\alpha-1} \sin(\alpha\pi) &= c \frac{1}{2} \sqrt{\frac{2}{\pi}} \left( \beta - \frac{1}{2} \right) r^{\beta-\frac{3}{2}} & \text{for } x_1 < 0 \\ 2a \sqrt{\frac{2}{\pi}} \alpha r^{\alpha-\frac{3}{2}} &= c \beta (\beta - 1) r^{\beta-2}. & \text{for } x_1 < 0. \end{aligned} \quad (57)$$

Since the exponents in  $r$  in the first equation in (57) must agree we see that  $\beta = \alpha + \frac{1}{2}$ . A simple calculation shows that (57) is only solvable if  $\alpha$  satisfies

$$\tan(\alpha\pi) = \frac{2}{\pi} \frac{\alpha}{\alpha^2 - \frac{1}{4}} \quad (58)$$

or

$$\alpha = \frac{1}{2} \quad (59)$$



**Figure 3:** Graph that shows the values of  $\alpha_k$  for  $k = 1, 2, 3, 4$ .

Given this we can conclude that if  $v$  and  $f$  are homogeneous solutions,  $v$  is even, then up to a multiplicative constant

$$(v, f) = \left( r^{\alpha_k} \cos(\alpha_k \phi), 2\sqrt{\frac{\pi}{2}} \sin(\alpha_k \pi) r^{\alpha_k + \frac{1}{2}} \right)$$

where  $\alpha_k$  is a solution to (58) or

$$v(r, \phi) = ar^{1/2} \cos(\phi/2) \text{ and } f(x_1) = 2\sqrt{\frac{\pi}{2}} ax_1.$$

The argument is similar if  $v$  is odd in  $x_2$ . In this case the equations involving  $f$  in (50) with  $(v, f) = (br^\alpha \sin(\alpha \phi), cr^\beta)$  reduces to

$$\begin{aligned} -b\alpha r^{\alpha-1} \cos(\alpha \pi) &= c \frac{1}{2} \sqrt{\frac{2}{\pi}} \left( \beta - \frac{1}{2} \right) r^{\beta-\frac{3}{2}} & \text{for } x_1 < 0 \\ 0 &= c\beta(\beta-1)r^{\beta-2} & \text{for } x_1 < 0. \end{aligned} \quad (60)$$

This implies that  $c = 0$  and thus that either  $a = 0$  or  $\cos(\alpha \pi) = 0$ . We may conclude that  $f = 0$  and that  $\alpha = k - \frac{1}{2}$ , for  $k \in \mathbb{N}$ , in case  $v$  is odd in  $x_2$ .  $\square$

## 4.2 The homogeneous solutions span $L^2(\partial B_1(0) \setminus (-1, 0))$ .

In this subsection we show that the set

$$\{1, \cos(\alpha_1 \phi), \cos(\alpha_2 \phi), \cos(\alpha_3 \phi), \dots\}$$

spans all the even functions in  $L^2((-\pi, \pi))$ , where  $\alpha_k > 1$  are the solutions to (56). Since  $\sin((k-1/2)\phi)$  spans all the odd functions in  $L^2((-\pi, \pi))$  it follows that for any  $g \in H^{1/2}(\partial B_1 \setminus (-1, 0))$  we can find a pair of solutions  $(v, f)$  to (50) where  $v = g$  on  $\partial B_1$  and is on the form

$$v(r, \phi) = a + \sum_{k=1}^{\infty} a_k r^{\alpha_k} \cos(\alpha_k \phi) + \sum_{k=1}^{\infty} b_k r^{k-1/2} \sin((k-1/2)\phi) \quad (61)$$

Notice that the expression of  $v$  in (61) does not contain the term  $a_0 r^{1/2} \cos(\phi/2)$  that appears in the form of  $v$  in Proposition 4.1. Later we will see that this missing term allows us to specify the values of  $f(-1)$ .

To show that  $\{\cos(\alpha_k \phi); k \in \mathbb{N}\}$  spans the set of even functions on the sphere we will show that this set has the same span as  $\{\cos(j\phi); j \in \mathbb{N}\}$ .

Consider the linear map  $A : L^2_{\text{even}}((-\pi, \pi)) \mapsto L^2_{\text{even}}((-\pi, \pi))$ , where we defined by

$$A \left( \sum_{k=0}^{\infty} a_k \cos(k\phi) \right) = \sum_{k=0}^{\infty} a_k \cos(\alpha_k \phi), \quad (62)$$

where  $L^2_{\text{even}}((-\pi, \pi))$  consists of the even functions of  $L^2((-\pi, \pi))$ .

We claim that  $\|A - I\|_{L^2_{\text{even}}((-\pi, \pi)) \mapsto L^2_{\text{even}}((-\pi, \pi))} < 1$ . From this it clearly follows that  $A$  is invertible and that  $\{\cos(\alpha_1 \phi), \cos(\alpha_2 \phi), \cos(\alpha_3 \phi), \dots\}$  spans  $L^2_{\text{even}}((-\pi, \pi))$ .



Notice that, for any even function  $u = \sum_{k=0}^{\infty} a_k \frac{\cos(k\phi)}{\sqrt{\pi}}$  the following estimate holds

$$\begin{aligned} \|(A - I)u\|_{L^2((-\pi, \pi))} &= \left\| \sum_{k=1}^{\infty} a_k \frac{\cos(\alpha_k \phi) - \cos(k\phi)}{\sqrt{\pi}} \right\|_{L^2} \leq \\ &\leq \sum_{k=1}^{\infty} |a_k| \left\| \frac{\cos(\alpha_k \phi) - \cos(k\phi)}{\sqrt{\pi}} \right\|_{L^2} \leq \\ &\leq \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \left( \sum_{k=1}^{\infty} \left\| \frac{\cos(\alpha_k \phi) - \cos(k\phi)}{\sqrt{\pi}} \right\|_{L^2}^2 \right)^{1/2} = \\ &= \|u\|_{L^2((-\pi, \pi))} \left( \sum_{k=1}^{\infty} \left\| \frac{\cos(\alpha_k \phi) - \cos(k\phi)}{\sqrt{\pi}} \right\|_{L^2}^2 \right)^{1/2}, \end{aligned}$$

where we used the triangle inequality and Hölder's inequality in as well as Paresval's equality. It is therefore enough to show that

$$\sum_{k=1}^{\infty} \left\| \frac{\cos(\alpha_k \phi) - \cos(k\phi)}{\sqrt{\pi}} \right\|_{L^2}^2 < 1.$$

We therefore estimate

$$\begin{aligned} &\frac{1}{\pi} \int_{-\pi}^{\pi} (\cos(\alpha_k \phi) - \cos(k\phi))^2 = \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left( 1 + \frac{\cos(2\alpha_k \phi)}{2} + \frac{\cos(2k\phi)}{2} - \cos((\alpha_k + k)\phi) - \cos((\alpha_k - k)\phi) \right) \leq \\ &\leq \frac{2}{\pi} \frac{\pi(\alpha_k - k) - \sin((\alpha_k - k)\pi)}{\alpha_k + k} + \frac{1}{\pi} \left( \frac{\sin(2\alpha_k \pi)}{2\alpha_k} - 2 \frac{\sin((\alpha_k + k)\pi)}{\alpha_k + k} \right) \leq \\ &\leq \frac{2}{\pi} \frac{\pi(\alpha_k - k) - \sin((\alpha_k - k)\pi)}{\alpha_k + k}, \end{aligned} \tag{63}$$

where we have used that  $k \leq \alpha_k < k + 1$  which implies that

$$\frac{1}{\pi} \left( \frac{\sin(2\alpha_k \pi)}{2\alpha_k} - 2 \frac{\sin((\alpha_k + k)\pi)}{\alpha_k + k} \right) \leq 0.$$

To estimate the term in (63) we need the elementary inequality

$$\pi\gamma_k - \frac{\pi^3 \gamma_k^3}{6} \leq \sin(\gamma_k \pi) \leq \pi\gamma_k. \tag{64}$$

Using the notation  $\alpha_k = k + \gamma_k$ , for  $0 < \gamma_k < 1$ , (63) can be estimated from above (using (64)) by

$$\frac{\pi^2}{6} \frac{\gamma_k^3}{k}.$$

In particular,

$$\sum_{k=1}^{\infty} \left\| \frac{\cos(\alpha_k \phi) - \cos(k\phi)}{\sqrt{\pi}} \right\|_{L^2}^2 \leq \sum_{k=1}^{\infty} \frac{\pi^2}{6} \frac{\gamma_k^3}{k}.$$

In order to show that this is less than one we need to know more about  $\gamma_k$ .

**Claim:**  $0 < \alpha_k - k < \frac{1}{4k}$ .

*Proof of the Claim:* Since

$$\tan(\pi\alpha_k) = \tan(\pi(\alpha_k - k)) \geq \pi(\alpha_k - k)$$

the solution  $\gamma_k$  to

$$\pi\gamma_k - \frac{2}{\pi} \frac{k + \gamma_k}{(k + \gamma_k)^2 - \frac{1}{4}} = 0, \quad (65)$$

will be larger than  $\alpha_k - k$ .

Rearranging the terms in (65) will lead to

$$(k\gamma_k) \left( (k\gamma_k)^2 + 2k^2(k\gamma_k) - \left( \frac{k^2}{4} + \frac{2k^2}{\pi^2} \right) \right) + \left( (k\gamma_k) - \frac{2}{\pi^2} \right) k^4 = 0. \quad (66)$$

We will define the function  $p(\cdot)$  so that  $p(k\gamma_k)$  takes the value to the left in (65), in particular equation (65) then states that  $p(k\gamma_k) = 0$ .

Since the function in (66) is increasing in  $k\gamma_k$  for  $k\gamma_k > 0$  if  $k \geq 1$  it is enough to show that  $p(1/4) > 0$  in order to deduce that the positive solution is less than  $1/4$ . A simple calculation gives

$$p(1/4) = \frac{1}{64} + \left( \frac{1}{4} - \frac{2}{\pi^2} \right) \left( k^4 + \frac{k^2}{4} \right) > 0.$$

Thus  $p(0) < 0 < p(1/4)$  and therefore, by the intermediate value property, there exists a solution  $p(k\gamma_k) = 0$  for  $0 < \gamma_k < \frac{1}{4k}$ . Since the function is increasing this is the only solution  $\gamma_k > 0$ . This proves the claim.

Using  $\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$  we may thus estimate

$$\|(A - I)u\|_{L^2((-\pi, \pi))} \leq \|u\|_{L^2} \sum_{k=1}^{\infty} \frac{\pi^2}{6} \frac{\gamma_k^3}{k} = \frac{\pi^6 \gamma^3}{6 \times 90} < 1.$$

We have therefore proved the following lemma.

**Lemma 4.2.** *The set*

$$\{1, \cos(\alpha_1\phi), \cos(\alpha_2\phi), \cos(\alpha_3\phi), \dots\}$$

where  $\alpha_k > 0$  are solutions to

$$\tan(\alpha\pi) = \frac{2}{\pi} \frac{\alpha}{\alpha^2 - \frac{1}{4}}$$

forms a basis of all even  $L^2(-\pi, \pi)$  functions.

### 4.3 Uniqueness of solutions.

**Lemma 4.3.** *The solutions to (50) that satisfy (51), (52) and (53) are unique.*

*Proof:* Clearly, by linearity, it is enough to show that solutions  $(v, f)$  such that  $v(x) = 0$  on  $\partial B_1(0)$  and  $f(-1) = 0$  are zero in the entire set  $B_1(0) \setminus \{(x_1, 0); x_1 \in (-1, 0)\}$  and on  $\{(x_1, 0); x_1 \in (-1, 0)\}$  respectively. Let  $(v, f)$  be such a solution. Then, with  $\Gamma_0 = \{(x_1, 0); x_1 \in (-1, 0)\}$

$$\begin{aligned} \int_{B_1(0) \setminus \Gamma_0} |\nabla v|^2 &= \int_{\partial B_1} v(x) \frac{\partial v(x)}{\partial \nu} + \\ &+ \int_{\Gamma_0} \left( v(x_1, 0^+) \frac{\partial v(x_1, 0^+)}{\partial \nu^+} + v(x_1, 0^-) \frac{\partial v(x_1, 0^-)}{\partial \nu^-} \right) = \\ &= \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_{\Gamma_0} v(x_1, 0^+) \frac{\partial}{\partial x_1} \left( \frac{1}{\sqrt{-x_1}} f(x_1) \right) + v(x_1, 0^-) \frac{\partial}{\partial x_1} \left( \frac{1}{\sqrt{-x_1}} f(x_1) \right) = \\ &= -\frac{1}{2} \sqrt{\frac{2}{\pi}} \int_{\Gamma_0} \left( \frac{\partial v(x_1, 0^+)}{\partial x_1} \frac{1}{\sqrt{-x_1}} f(x_1) + \frac{\partial v(x_1, 0^-)}{\partial x_1} \frac{1}{\sqrt{-x_1}} f(x_1) \right) = \\ &= \frac{1}{2} \int_{\Gamma_0} \frac{\partial^2 f(x_1)}{\partial x_1^2} f(x_1) = -\frac{1}{2} \int_{\Gamma_0} |f'(x_1)|^2, \end{aligned}$$

where we used integration by parts several times and that the appearing boundary terms in these integration by parts vanish due to (52) and (53) finally we also used the relations between  $v$  and  $f$  on  $\Gamma_0$  specified in (50). Notice that the left hand side is non-negative and the right hand side is non-positive. We may conclude that both the left and right hand side are zero.

We have thus proved that if  $v$  and  $f$  vanish on the boundary then

$$\int_{B_1(0) \setminus \Gamma_0} |\nabla v|^2 + \frac{1}{2} \int_{\Gamma_0} |f'(x_1)|^2 = 0.$$

□

#### 4.4 Existence of solutions.

**Lemma 4.4.** *Given  $g \in H^{1/2}(\partial B_1(0) \setminus \{(-1, 0)\})$  and  $t \in \mathbb{R}$  there exists a solution to (50) satisfying (51), (52) and (53).*

*Proof:* Since

$$\{1, \cos(\alpha_1 \phi), \cos(\alpha_2 \phi), \cos(\alpha_3 \phi), \dots\} \quad (67)$$

span the even functions on  $L^2(-\pi, \pi)$  and

$$\{\sin(\phi/2), \sin(3\phi/2), \sin(5\phi/2), \dots\}$$

span the odd functions on  $L^2(-\pi, \pi)$  we may express any function  $g \in H^{1/2}(\partial B_1(0) \setminus (-1, 0))$  according to

$$g = a + \sum_{k=1}^{\infty} a_k \cos(\alpha_k \phi) + \sum_{k=1}^{\infty} b_k \sin((k - 1/2)\phi)$$

clearly

$$u(r, \phi) = a + \sum_{k=1}^{\infty} a_k r^{\alpha_k} \cos(\alpha_k \phi) + \sum_{k=1}^{\infty} b_k r^{k-1/2} \sin((k - 1/2)\phi)$$

and

$$f(x_1) = \sum_{k=1}^{\infty} a_k 2\sqrt{\frac{\pi}{2}} \sin(\alpha_k \pi) |x_1|^{\alpha_k + \frac{1}{2}}$$

will solve (50) except possible the condition that  $f(-1) = t$ .

However, since we may expand  $r^{1/2} \cos(\phi/2)$  in the basis (67) on the boundary  $\partial B_1(0)$  we can find a solution

$$w(r, \phi) = b + r^{1/2} \cos(\phi/2) + \sum_{k=1}^{\infty} b_k r^{\alpha_k} \cos(\alpha_k \phi)$$

$$h(x_1) = 2\sqrt{\frac{\pi}{2}} x_1 + \sum_{k=1}^{\infty} b_k 2\sqrt{\frac{\pi}{2}} \sin(\alpha_k \pi) |x_1|^{\alpha_k + \frac{1}{2}}$$

such that  $w = 0$  on  $\partial B_1(0)$ . Clearly  $h(-1) \neq 0$  since if  $h(-1) = 0$  then  $u = h = 0$  by the uniqueness of solutions. It follows that

$$\left( u + \frac{t - f(-1)}{h(-1)} w, f(x_1) + \frac{t - f(-1)}{h(-1)} h(x_1) \right)$$

is a solution that satisfies (50).  $\square$

#### 4.5 Regularity of the Solution to the Linearized problem.

The following is a simple Corollary to Proposition 4.1.

**Corollary 4.1.** *Let  $(v, f)$  be a solution to (50) of the following form*

$$v(r, \phi) = a + \sum_{k=1}^{\infty} a_k r^{\alpha_k} \cos(\alpha_k \phi) + \sum_{k=2}^{\infty} b_k r^{k-1/2} \sin((k-1/2)\phi)$$

and

$$f(x_1) = \sum_{k=1}^{\infty} a_k 2\sqrt{\frac{\pi}{2}} \sin(\alpha_k \pi) |x_1|^{\alpha_k + \frac{1}{2}},$$

that is  $a_0 = b_1 = 0$  in the expression in Proposition 4.1. Then for each  $\alpha < \alpha_1$  there exists an  $s_\alpha > 0$ , depending only on  $\alpha$ , such that

$$\|\nabla v\|_{L^2(B_{s_\alpha}(0) \setminus \Gamma_0)} < s_\alpha^\alpha \|\nabla v\|_{L^2(B_1 \setminus \Gamma_0)} \quad (68)$$

and

$$\|f'(x_1)\|_{L^2(-s_\alpha, 0)} \leq s_\alpha^{\alpha-1/2} \|f'\|_{L^2(-1, 0)}. \quad (69)$$

Furthermore if  $a_1 = 0$  then the same is true for each  $\alpha < \alpha_2$ .

*Proof:* We will only show (68), the proof of (69) is similar (and somewhat simpler). Notice that

$$\begin{aligned} \|\nabla v\|_{L^2(B_{s_\alpha}(0) \setminus \Gamma_0)} &\leq \sum_{k=1}^{\infty} |a_k| \|\nabla r^{\alpha_k} \cos(\alpha_k \phi)\|_{L^2(B_{s_\alpha}(0) \setminus \Gamma_0)} + \\ &\sum_{k=2}^{\infty} |b_k| \left\| \nabla r^{k-1/2} \sin((k-1/2)\phi) \right\|_{L^2(B_{s_\alpha}(0) \setminus \Gamma_0)}. \end{aligned} \quad (70)$$

A change of variables  $r \mapsto s_\alpha r$  shows that

$$\begin{aligned} \|\nabla r^{\alpha_k} \cos(\alpha_k \phi)\|_{L^2(B_{s_\alpha}(0) \setminus \Gamma_0)} &= \left( \int_{B_{s_\alpha} \setminus \Gamma_0} |\nabla r^{\alpha_k} \cos(\alpha_k \phi)|^2 \right)^{1/2} = \\ &= s_\alpha^{\alpha_k} \left( \int_{B_1 \setminus \Gamma_0} |\nabla r^{\alpha_k} \cos(\alpha_k \phi)|^2 \right)^{1/2}, \end{aligned}$$

a similar calculation obviously works for the  $r^{k-1/2} \sin((k-1/2)\phi)$  terms in (70).

We may thus write (70)

$$\begin{aligned} \|\nabla v\|_{L^2(B_{s_\alpha}(0) \setminus \Gamma_0)} &\leq \sum_{k=1} s_\alpha^{\alpha_k} |a_k| \|\nabla r^{\alpha_k} \cos(\alpha_k \phi)\|_{L^2(B_1(0) \setminus \Gamma_0)} + \\ &\sum_{k=2}^\infty |b_k| s_\alpha^{k-1/2} \left\| \nabla r^{k-1/2} \sin((k-1/2)\phi) \right\|_{L^2(B_1(0) \setminus \Gamma_0)} \leq \quad (71) \\ &\leq s_\alpha^{\alpha_1} \left( \sum_{k=1} |a_k| \|\nabla r^{\alpha_k} \cos(\alpha_k \phi)\|_{L^2(B_1(0) \setminus \Gamma_0)} + \right. \\ &\left. + \sum_{k=2}^\infty |b_k| \left\| \nabla r^{k-1/2} \sin((k-1/2)\phi) \right\|_{L^2(B_1(0) \setminus \Gamma_0)} \right). \end{aligned}$$

We already showed, in sub-section 4.2, that we may estimate (71) by

$$\begin{aligned} &\leq (C s_\alpha^{\alpha_1 - \alpha}) s_\alpha^\alpha \left( \sum_{k=1} |a_k| \left\| \sum_{k=1} a_k \nabla r^{\alpha_k} \cos(\alpha_k \phi) \right\|_{L^2(B_1(0) \setminus \Gamma_0)} + \right. \\ &\left. + \left\| \sum_{k=2}^\infty b_k \nabla r^{k-1/2} \sin((k-1/2)\phi) \right\|_{L^2(B_1(0) \setminus \Gamma_0)} \right), \end{aligned}$$

for some constant  $C$  depending on the almost orthogonality of the basis  $\cos(\alpha_k \phi)$ . Choosing  $s_\alpha$  small enough so that

$$C s_\alpha^{\alpha_1 - \alpha} < 1$$

finishes the proof.  $\square$

## 5 Strong Convergence.

In this section we prove that the linearizing sequence  $v^j$  and  $f_j$  converges strongly in  $W^{1,2}$ . Throughout this section  $u^j$ ,  $\Gamma_{u^j}$ ,  $v^j$  and  $f_j$  will be as in Proposition 3.1.

We begin this section with a lemma that proves strong convergence under an extra assumption.

**Lemma 5.1.** *Let  $u^j$ ,  $\Gamma_{u^j}$ ,  $v^j$  and  $f_j$  be as in Proposition 3.1. Furthermore assume that there exists a constant  $C$  such that*

$$\left\| \nabla \left( \frac{u^j(rx)}{\sqrt{r}} - \Pi \left( \frac{u^j(rx)}{\sqrt{r}} \right) \right) \right\|_{L^2(B_1(0) \setminus \Gamma_{u^j_r})} \leq C\epsilon_j \quad (72)$$

for all  $j$ .

Then

1.  $\text{Reg}(\nabla v^j) \rightarrow \text{Reg}(\nabla v^0)$  strongly in  $L^2(B_1)$ .
2. and  $v^0$  and  $f_0$  satisfies the following estimates

$$\|\nabla v^0\|_{B_r(0) \setminus \Gamma_0} \leq C(1 + \ln(1/r))\sqrt{r}, \quad (73)$$

$$|f_0(x_1)| \leq C(1 + \ln(1/|x_1|))|x_1| \quad (74)$$

and

$$|f'_0(x_1)| \leq C(1 + \ln(1/|x_1|)). \quad (75)$$

*Proof:* The proof of the strong convergence is based on the general fact that if  $\|w^j\|_{L^2(\Omega)} \leq Cr^{1/2}$  and  $w^j \rightarrow w^0$  weakly in  $L^2(\Omega)$  then

$$\lim_{j \rightarrow \infty} \|w^j - w^0\|_{L^2(\Omega)}^2 = \lim_{j \rightarrow \infty} \|w^j\|_{L^2(\Omega)}^2 - \|w^0\|_{L^2(\Omega)}^2 \leq C^2 r. \quad (76)$$

In this proof we let

$$\epsilon_j w_r^j(x) = u^j(x) - \Pi(u^j, r)$$

then, by assumption (72),  $\|\nabla w_r^j\|_{L^2(B_1 \setminus \Gamma_{u^j_r})} \leq C\sqrt{r}$ . Also

$$\begin{aligned} v^j - w_r^j &= \Pi(u^j, 1) - \Pi(u^j, r) = r^{1/2} (a_j \sin(\phi/2) + b_j \cos(\phi/2)) \rightarrow \\ &\rightarrow r^{1/2} (a \sin(\phi/2) + b \cos(\phi/2)) \text{ as } j \rightarrow \infty, \end{aligned}$$

since both  $\Pi(u^j, 1)$  and  $\Pi(u^j, r)$  are trigonometric functions of the same form as the right hand side.

Clearly,  $w_r^j \rightarrow v^0 + r^{1/2} (a \sin(\phi/2) + b \cos(\phi/2))$  weakly in  $B_r(0) \setminus \Gamma_{u^j}$  and

$$\begin{aligned} &\|\nabla (v^j - v^0)\|_{L^2(B_r \setminus \Gamma_{u^j})} = \\ &= \left\| \nabla \left( w_r^j - \left( v^0 + r^{1/2} (a \sin(\phi/2) + b \cos(\phi/2)) \right) \right) \right\|_{L^2(B_r \setminus \Gamma_{u^j})} \end{aligned}$$

but the limit of the right side of the above equation will be bounded by  $2Cr^{1/2}$  for large  $j$  by (76) since  $\|\nabla w_r^j\|_{L^2(B_1 \setminus \Gamma_{u^j_r})} \leq C\sqrt{r}$ .

This implies that

$$\begin{aligned} &\lim_{j \rightarrow \infty} \|\nabla (v^j - v^0)\|_{L^2(B_1(0) \setminus \Gamma_{u^j})}^2 = \\ &= \lim_{j \rightarrow \infty} \|\nabla (v^j - v^0)\|_{L^2(B_1(0) \setminus (B_r(0) \cup \Gamma_{u^j}))}^2 + \end{aligned}$$

$$+ \lim_{j \rightarrow \infty} \|\nabla (v^j - v^0)\|_{L^2(B_r(0) \setminus \Gamma_{u^j})}^2 \leq Cr^{1/2}$$

since  $\text{Reg}(\nabla v^j) \rightarrow \text{Reg}(\nabla v^0)$  strongly in  $B_1 \setminus B_r$ . Since we can choose  $r$  arbitrarily small it follows that

$$\lim_{j \rightarrow \infty} \|\nabla (v^j - v^0)\|_{L^2(B_1(0) \setminus \Gamma_{u^j})}^2 = 0,$$

which is the strong convergence we claim.

To see that  $v^0$  satisfies the required estimates we notice that

$$\Pi(u^j, 2^{-k}) = \Pi(\Pi(u^j, 2^{-k+1}) + \epsilon_j w_{2^{-k+1}}^j, 1/2)$$

from which it follows that

$$\|\nabla (\Pi(u^j, 2^{-k}) - \Pi(u^j, 2^{-k+1}))\|_{L^2(B_1)} \leq \|\nabla w_{2^{-k+1}}^j\|_{L^2(B_1)} \leq C\epsilon_j.$$

It follows that

$$\begin{aligned} & \|\nabla (\Pi(u^j, 1) - \Pi(u^j, 2^{-k}))\|_{L^2(B_1)} \leq \\ & \leq \sum_{l=0}^{k-1} \|\nabla (\Pi(u^j, 2^{-l}) - \Pi(u^j, 2^{-l+1}))\|_{L^2(B_1)} \leq Ck\epsilon_j. \end{aligned}$$

We can thus conclude that, with  $r = 2^{-k}$ ,

$$\begin{aligned} \|\nabla v^j\|_{L^2(B_r(0) \setminus \Gamma_{u^j})} &= \left\| \nabla \left( w_r^j + \frac{1}{\epsilon_j} (\Pi(u^j, 1) - \Pi(u^j, r)) \right) \right\|_{L^2(B_r(0) \setminus \Gamma_{u^j})} \leq \\ &\leq \|\nabla w_r^j\|_{L^2(B_r(0) \setminus \Gamma_{u^j})} + \frac{1}{\epsilon_j} \|\nabla (\Pi(u^j, 1) - \Pi(u^j, r))\|_{L^2(B_r(0) \setminus \Gamma_{u^j})} \leq \\ &\leq C\sqrt{r} + Ck\sqrt{r} = C(1 + \ln(1/r))\sqrt{r}. \end{aligned}$$

Passing to the limit  $j \rightarrow \infty$  implies the desired estimate for  $v^0$ .

To derive the desired estimates for  $f_0$  we notice that if

$$\Pi(u, r) = \lambda_r r^{1,2} \sin\left(\frac{\phi - \phi_r}{2}\right)$$

then

$$\|\nabla (\Pi(u^j, 1) - \Pi(u^j, 2^{-k}))\|_{L^2(B_1)} \leq Ck2^{-k}\epsilon_j.$$

This implies, using a Taylor expansion, that if  $\epsilon_j$  is small then

$$|\phi_1 - \phi_{2^{-k}}| \leq Ck2^{-k}\epsilon_j. \quad (77)$$

In order to prove (74) we just notice that rotating the coordinate system by an angle  $\phi_k$  amounts to subtracting a linear function  $l(x_1) = a_k x_1$  from  $f_j$ , modulo lower order terms, where  $\epsilon_j a_k \approx \phi_k + O((\phi_k)^3)$  for  $\phi_k$  small enough (that which follows from  $\epsilon_j$  being small). In particular  $|a_k| \leq C$ .

From Corollary 2.2 we may conclude that

$$\sup_{B_1(0) \setminus B_{1/2}, x_1 < 0} \left| \frac{\epsilon_j f_j(2^{-k} x_1)}{2^{-k}} - a_k x_1 \right| \leq C k \epsilon_j. \quad (78)$$

But (78), together with  $|a_k| \leq Ck$ , implies that

$$-C \ln(1/|x_1|)|x_1| \leq f(x_1) \leq C \ln(1/|x_1|)|x_1|.$$

To prove (75) one argues similarly. In particular, Lemma 2.3, together with (78) and that  $u^j$  is  $C\epsilon_j \sqrt{\ln(1/r)}$  close to a crack-tip (by 73) will imply that  $|f'(x_1) - a_k| \leq C \ln(1/|x_1|)$  for any  $x_1 \in (-2^{-k}, -2^{-k-1})$ . We leave the details to the reader.  $\square$

**Lemma 5.2.** *Let  $u^j$ ,  $\Gamma_{u^j}$  and  $v^j$  be as in Proposition 3.1. Then*

$$\lim_{j \rightarrow \infty} \sup_{r \in (0,1]} \frac{1}{\sqrt{r}} \|\nabla(u^j - \Pi(u^j, r))\|_{L^2(B_r \setminus \Gamma_{u^j})} = 0 \quad (79)$$

and for each  $j$  the supremum is achieved at some  $r_j \in (0, 1]$ .

*Proof:* First we notice that since for every  $j$  any subsequence  $r \rightarrow 0$

$$\lim_{r \rightarrow 0} \frac{u^j(rx)}{\sqrt{r}} = \sqrt{\frac{2}{\pi}} r^{1/2} \sin((\phi + \phi_0)/2),$$

that is, for every  $j$ ,

$$\lim_{r \rightarrow 0} \frac{1}{\sqrt{r}} \|\nabla(u^j - \Pi(u^j, r))\|_{L^2(B_r \setminus \Gamma_{u^j})} = 0.$$

It follows that any positive supremum of

$$\frac{1}{\sqrt{r}} \|\nabla(u^j - \Pi(u^j, r))\|_{L^2(B_r \setminus \Gamma_{u^j})}$$

occurs at a strictly positive  $r$ .

To see that (79) holds we argue by contradiction and assume that there exists a  $\delta > 0$  and  $r_j > 0$  such that the supremum is achieved at  $r_j$  and

$$\frac{1}{\sqrt{r_j}} \|\nabla(u^j - \Pi(u^j, r))\|_{L^2(B_{r_j} \setminus \Gamma_{u^j})} \geq \delta > 0.$$

We also notice that, by Lemma 2.1 and the remark thereafter,

$$\Pi(u^j, 1) \rightarrow \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi) \Rightarrow \int_{B_1 \setminus \Gamma_{u^j}} |\nabla u^j|^2 \rightarrow 1 \quad (80)$$

and, by Lemma 2.4 and the remark after Lemma 2.1,

$$\lim_{r \rightarrow 0} \int_{B_1 \setminus \Gamma_{u^j}} \left| \nabla \frac{u^j(rx)}{\sqrt{r}} \right|^2 \rightarrow 1. \quad (81)$$



We may conclude that  $\frac{u^j(r_j x)}{\sqrt{r_j}} \rightarrow u^0$  where  $u^0$  is a minimizer of the Mumford-Shah functional,

$$\|\nabla(u^0 - \Pi(u^0, 1))\|_{L^2(B_1 \setminus \Gamma_{u^0})} \geq \delta > 0 \quad (82)$$

and

$$1 = \lim_{r \rightarrow 0} \frac{1}{r} \int_{B_r(0) \setminus \Gamma_{u^0}} |\nabla u^0|^2 \leq \frac{1}{r} \int_{B_r(0) \setminus \Gamma_{u^0}} |\nabla u^0|^2 \leq \int_{B_1(0) \setminus \Gamma_{u^0}} |\nabla u^0|^2 = 1 \quad (83)$$

where we used (80) and the Monotonicity formula (Lemma 2.4) in the last step.

The monotonicity formula (Lemma 2.4) together with (83) implies that  $u^0$  is homogeneous which implies that

$$u^0 = \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi)$$

which contradicts (82).

We can conclude that

$$\lim_{j \rightarrow \infty} \sup_{r \in (0, 1]} \frac{1}{\sqrt{r}} \|\nabla(u^j - \Pi(u^j, r))\|_{L^2(B_r \setminus \Gamma_{u^j})} = 0.$$

□

**Proposition 5.1.** *Let  $u^j$  and  $\Gamma_{u^j}$  be as in Proposition 3.1.*

*There exists a constant  $C$  such that*

$$\left\| \nabla \left( \frac{u^j(rx)}{\sqrt{r}} - \Pi \left( \frac{u^j(rx)}{\sqrt{r}}, 1 \right) \right) \right\|_{L^2(B_1(0) \setminus \Gamma_{u^j_r})} \leq C \epsilon_j. \quad (84)$$

*In particular,  $u^0$  and  $f_0$ , satisfies the estimates in Lemma 5.1.*

*Proof:* We will again argue by contradiction and assume that there is a sequence  $(u^j, f_j)$  that are  $\epsilon_j \rightarrow 0$  close to a crack-tip such that

$$\left\| \nabla \left( \frac{u^j(\tilde{r}_j x)}{\sqrt{\tilde{r}_j}} - \Pi \left( \frac{u^j(\tilde{r}_j x)}{\sqrt{\tilde{r}_j}}, 1 \right) \right) \right\|_{L^2(B_1(0) \setminus \Gamma_{u^j_{\tilde{r}_j}})} = j \epsilon_j. \quad (85)$$

for some sequence  $\tilde{r}_j \in (0, 1]$  such that the expression on the right in (85) is maximized for  $\tilde{r}_j$ . By Lemma 5.2 it follows that  $j \epsilon_j \rightarrow 0$  as  $j \rightarrow \infty$ . It is also easy to see that  $\tilde{r}_j \rightarrow 0$ .

We let  $r_j = c \tilde{r}_j$  where  $c > 1$  is some fixed constant that depends on  $\alpha_1$  and the norm of the mapping  $A - I$  introduced in subsection 4.2.

We define

$$u^j_{r_j}(x) = \frac{u^j(r_j x)}{\sqrt{r_j}}.$$

Then  $u^j_{r_j}$  satisfies the criteria in Lemma 5.1 and the sequence

$$v^j(x) = \frac{\frac{u^j(r_j x)}{\sqrt{r_j}} - \Pi\left(\frac{u^j(r_j x)}{\sqrt{r_j}}, 1\right)}{j \epsilon_j}$$

converges strongly to a solution of Proposition 4.1.

In particular,  $v^0$  satisfies the series expansions in (54)

$$v^0(r, \phi) = a + a_0 r^{1/2} \cos(\phi/2) + \sum_{k=1}^{\infty} a_k r^{\alpha_k} \cos(\alpha_k \phi) + \sum_{k=1}^{\infty} b_k r^{k-1/2} \sin((k-1/2)\phi) \quad (86)$$

and (55)

$$f_0(x_1) = 2a_0 \sqrt{\frac{\pi}{2}} x_1 + \sum_{k=1}^{\infty} a_k 2 \sqrt{\frac{\pi}{2}} \sin(\alpha_k \pi) |x_1|^{\alpha_k + \frac{1}{2}}.$$

But by our choice of  $r_j = c\tilde{r}_j$ , and that the maximum occurs at  $\tilde{r}_j$ , it follows that

$$\begin{aligned} & \left\| \nabla \left( \frac{u^j(r_j x)}{\sqrt{r_j}} - \Pi \left( \frac{u^j(r_j x)}{\sqrt{r}} \right) \right) \right\|_{L^2(B_1(0) \setminus \Gamma_{u_{r_j}^j})} \leq \\ & \leq \left\| \nabla \left( \frac{u^j(\tilde{r}_j x)}{\sqrt{\tilde{r}_j}} - \Pi \left( \frac{u^j(\tilde{r}_j x)}{\sqrt{\tilde{r}_j}} \right) \right) \right\|_{L^2(B_1(0) \setminus \Gamma_{u_{\tilde{r}_j}^j})} \end{aligned}$$

which implies that

$$\|\nabla(v^0 - \Pi(v^0, 1))\|_{L^2(B_1 \setminus \Gamma_0)} \leq \|\nabla(\sqrt{c}v^0(x/c) - \Pi(\sqrt{c}v^0(x/c), 1))\|_{L^2(B_1 \setminus \Gamma_0)}$$

which is not true if we choose  $c$  small enough (since all the terms in (86) have homogeneity greater than  $1/2$ ). This is a contradiction. We may conclude that there exists a constant  $C$  such that (84) holds.

By the inequality (84) and Lemma 5.1 the second conclusion holds.  $\square$

## 6 $C^{1,\alpha}$ -regularity of the Crack-Tip.

In this section we prove the first main result. We begin by a simple Lemma that states that we may rotate the coordinate system to get rid of the  $\cos(\phi/2)$  term in the asymptotic expansion of  $v^0$ . The Lemma is a somewhat annoying technical curiosity. The need for this lemma arises because we define  $v^j$  in such a way that  $v^j$  is orthogonal to  $\sin(\phi/2)$  and  $\cos(\phi/2)$  - which is natural when we consider convergence properties of  $v^j$ . But for the regularity theory we would want  $v^0$  to consist of terms of higher homogeneities than  $1/2$ . But the even terms,  $r^{\alpha_k} \cos(\alpha_k \phi)$ , in the homogeneous expansion of  $v^0$  are not orthogonal to  $r^{1/2} \cos(\phi/2)$ . These non-orthogonality properties means that we get an extra  $a_0 r^{1/2} \cos(\phi/2)$  term in the homogeneous expansion of the solutions to the linearized problem in Proposition 4.1. Fortunately it is rather easy to get rid of this extra term by slightly rotating the coordinate system.

**Lemma 6.1.** *Let  $(u^j, \Gamma_j)$  be as in Proposition 3.1 (In particular Proposition 5.1 holds). Then there exists a sequence of rotations of  $\mathbb{R}^2$ ,  $P^j : \mathbb{R}^2 \mapsto \mathbb{R}^2$  such that if we express  $(u^j, \Gamma_u^j)$  in these rotated coordinate systems and define  $v^j \rightarrow v^0$  and  $f_j \rightarrow f_0$  in these rotated coordinate systems then*

$$v^0(r, \phi) = a + \sum_{k=1}^{\infty} a_k r^{\alpha_k} \cos(\alpha_k \phi) + \sum_{k=1}^{\infty} b_k r^{k-1/2} \sin((k-1/2)\phi) \quad (87)$$

and

$$f_0(x_1) = \sum_{k=1}^{\infty} a_k 2\sqrt{\frac{\pi}{2}} \sin(\alpha_k \pi) |x_1|^{\alpha_k + \frac{1}{2}}. \quad (88)$$

*Proof:* We choose  $P^j$  to be the rotation of an angle  $c\epsilon_j$ ,  $P^j(r, \phi) = (r, \phi + c\epsilon_j)$ . We may apply Proposition 3.1 on the sequence of rotated solutions  $(u^j(P^j(r, \phi)), P^j(\Gamma_{u^j}))$ . By Proposition 3.1 and Proposition 5.1 it follows that the convergence  $v^j \rightarrow v^0$  and  $f_j \rightarrow f_0$  is strong and  $(v^0, f_0)$  will satisfy the estimates needed to apply the second half of Proposition 4.1. Therefore by Proposition 4.1  $v^0$  and  $f_0$  will be of the following form

$$v^0(r, \phi) = a + a_0 r^{1/2} \cos(\phi/2) + \sum_{k=1}^{\infty} a_k r^{\alpha_k} \cos(\alpha_k \phi) + \sum_{k=1}^{\infty} b_k r^{k-1/2} \sin((k-1/2)\phi)$$

and

$$f_0(x_1) = a_0 2\sqrt{\frac{\pi}{2}} x_1 + \sum_{k=1}^{\infty} a_k 2\sqrt{\frac{\pi}{2}} \sin(\alpha_k \pi) |x_1|^{\alpha_k + \frac{1}{2}}$$

We need to show that, if we choose the rotations appropriately, then  $a_0 = 0$ . It is easy to see that slightly rotate the coordinate system by  $c\epsilon_j$  amounts to adding  $cx_1$  to the limit function  $f_0$ . Thus by choosing  $c$  appropriately we will get  $a_0 = 0$ . The argument is not very illustrative. But we provide the details for the sake of completeness.

Remember that  $u^j(r, \phi) = \Pi(u^j) + \epsilon_j v^j(r, \phi)$  and that, for some  $\lambda_j \in \mathbb{R}$ ,

$$\Pi(u^j)(r, \phi) = \sqrt{\frac{2}{\pi}} \lambda_j \sin\left(\frac{\phi}{2}\right).$$

This implies that

$$\begin{aligned} u^j(r, \phi + c\epsilon_j) &= \Pi(u^j)(r, \phi + c\epsilon_j) + \epsilon_j v^j(r, \phi + c\epsilon_j) = \\ &= \sqrt{\frac{2}{\pi}} \lambda_j \left( \sin\left(\frac{\phi}{2}\right) \cos\left(\frac{c\epsilon_j}{2}\right) + \cos\left(\frac{\phi}{2}\right) \sin\left(\frac{c\epsilon_j}{2}\right) \right) + \epsilon_j v^j(r, \phi + c\epsilon_j) = \\ &= \sqrt{\frac{2}{\pi}} \left[ \lambda_j \cos\left(\frac{c\epsilon_j}{2}\right) \right] \sin\left(\frac{\phi}{2}\right) + \epsilon_j \left[ \cos\left(\frac{\phi}{2}\right) \frac{\sin\left(\frac{c\epsilon_j}{2}\right)}{\epsilon_j} + v^j(r, \phi + c\epsilon_j) \right]. \end{aligned} \quad (89)$$

If we denote the first square bracket in (89) by  $\tilde{\lambda}_j$  and the second square bracket  $\tilde{v}^j$  then we see that if  $c$  is chosen so that

$$\lim_{j \rightarrow \infty} \int_{B_1(0) \setminus \Gamma_{u^j}} \nabla \tilde{v}^j \cdot \nabla \left( r^{1/2} \cos(\phi/2) \right) = 0$$

then it follows that  $\tilde{v}^j \rightarrow \tilde{v}^0$  where  $\tilde{v}^0$  has the form in (87). Since  $(\tilde{v}^0, \tilde{f}_0)$  will satisfy the equations (50) it follows that  $\tilde{f}_0$  has the form of (88), see Proposition 4.1.  $\square$

**Theorem 6.1.** *For every  $\alpha \leq \alpha_1$  there exists an  $\epsilon_\alpha > 0$  such that if  $(u, \Gamma)$  is  $\epsilon$ -close to a crack-tip solution for some  $\epsilon < \epsilon_\alpha$ . Then, with  $s_\alpha$  as in Corollary 4.1,  $(u_{s_\alpha}, \Gamma_{u_{s_\alpha}})$  is  $s_\alpha^{\alpha-1/2} \epsilon$ -close to a crack-tip solution:*

$$\left\| \nabla \left( \frac{u(s_\alpha x)}{\sqrt{s_\alpha}} - \Pi \left( \frac{u(s_\alpha x)}{\sqrt{s_\alpha}} \right) \right) \right\|_{L^2(B_1 \setminus \Gamma_{u_{s_\alpha}})} \leq s_\alpha^{\alpha-1/2} \|\nabla(u - \Pi(u))\|_{L^2(B_1 \setminus \Gamma_u)}.$$

*Proof:* The proof is almost trivial. We argue by contradiction and assume that there exists a sequence of minimizers  $(u^j, \Gamma_{u^j})$  that are  $\epsilon_j \rightarrow 0$  close to a crack-tip solution but  $(u_{s_\alpha}^j, \Gamma_{u_{s_\alpha}^j})$  is not  $s_\alpha^{\alpha-1/2}\epsilon$ -close to a crack-tip solution.

Denoting, as in Proposition 3.1,

$$v^j = \frac{u^j - \Pi(u^j)}{\epsilon_j}$$

we see that the assumption in the previous paragraph implies that

$$\|\nabla v^j\|_{L^2(B_{s_\alpha} \setminus \Gamma_{u^j})} \geq s_\alpha^\alpha. \quad (90)$$

If we use appropriately rotated coordinates, then by Lemma 6.1,  $v^j \rightarrow v^0$  strongly (by Proposition 5.1) where  $v^0$  solves the linear system in Proposition 4.1 with  $a_0 = 0$  and thus by Corollary 4.1

$$\|\nabla v^0\|_{L^2(B_{s_\alpha} \setminus \Gamma_0)} < s_\alpha^\alpha \|\nabla v^0\|_{L^2(B_1 \setminus \Gamma_0)} = s_\alpha^\alpha.$$

This contradicts the strong convergence and (90). This finishes the proof.  $\square$

**Corollary 6.1.** *There exists an  $\epsilon_0 > 0$  such that if  $(u, \Gamma)$  is  $\epsilon$ -close to a crack-tip solution for some  $\epsilon \leq \epsilon_0$  then  $\Gamma_u$  is  $C^{1,\alpha}$  at the crack-tip for every  $\alpha < \alpha_1 - 1/2$ .*

*This in the sense that the tangent at the crack-tip is a well defined line, which we may assume to be  $\{(x_1, 0); x_1 \in \mathbb{R}\}$ , and there exists a constant  $C_\alpha$  such that*

$$\Gamma_u \subset \{(x_1, x_2); |x_2| < C_\alpha \epsilon |x_1|^{1+\alpha}, x_1 < 0\}.$$

Here  $C_\alpha$  may depend on  $\alpha$  but not on  $\epsilon < \epsilon_0$ .

*Proof:* Denoting  $t = s_\alpha$  it follows directly from an iteration of Theorem 6.1 that if  $\epsilon$  is small enough then

$$\|\nabla(u_{t^k} - \Pi(u_{t^k}))\|_{L^2(B_1 \setminus \Gamma_{u_{t^k}})} \leq t^{(\alpha-1/2)k} \epsilon. \quad (91)$$

Since, by the triangle inequality

$$\|\nabla(\Pi(u_{t^{k+1}}) - \Pi(u_{t^k}))\|_{L^2(B_1 \setminus \Gamma_0)} \leq C \|\nabla(u_{t^k} - \Pi(u_{t^k}))\|_{L^2(B_1 \setminus \Gamma_{u_{t^k}})} \quad (92)$$

for some dimensional constant we see that  $\Pi(u_{t^k})$  forms a Cauchy sequence in  $W^{1,2}$  and thus  $\lim_{k \rightarrow \infty} \Pi(u_{t^k}) = \Pi_0$  exists. By a choice of coordinate system we may assume that

$$\Pi_0 = \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2). \quad (93)$$

Also, from the triangle inequality, (92) and (91)

$$\|\nabla(\Pi(u_{t^k}) - \Pi_0)\|_{L^2} \leq C \epsilon \sum_{j=k}^{\infty} t^{\alpha j} \leq C \epsilon t^{(\alpha-1/2)k}.$$

From Corollary 2.2 we can also conclude that the free discontinuity set  $\Gamma_{u_{t^k}}$  is within distance  $C \epsilon t^{(\alpha-1/2)k}$  from a line  $l_{t^k} = \{(x_1, x_2); x_2 = a_k x_1\}$ .

The coefficients  $a_k$  and  $a_{k+1}$  will differ by a less than  $C\epsilon t^{(\alpha-1)k}$  for some constant  $C$ . This since

$$\begin{aligned} & \left\{ x; x \in B_{t^{k+1}}, \text{dist}(x, l_{t^{k+1}}) \leq C\epsilon t^{(\alpha-\frac{1}{2})(k+1)} \right\} \subset \\ & \subset \left\{ x; x \in B_{t^k}, \text{dist}(x, l_{t^k}) \leq C\epsilon t^{(\alpha-\frac{1}{2})k} \right\} \end{aligned}$$

only if

$$|a_k - a_{k+1}| \leq C\epsilon t^{(\alpha-1/2)k} \quad (94)$$

The coefficients  $a_k$  therefore forms a Cauchy sequence and therefore converges. By the rotation made in (93) we may conclude that  $a_k \rightarrow 0$ . Moreover, by the triangle inequality and (94)

$$|a_k| \leq \sum_{j=k}^{\infty} |a_j - a_{j+1}| \leq C\epsilon t^{(\alpha-1/2)k}. \quad (95)$$

It follows from (95) and Corollary 2.2 that

$$\Gamma_k \cap (B_{t^k}(0) \cap B_{t^k}(0)) \subset \{(x_1, x_2); |x_2| \leq C\epsilon t^{(\alpha+1/2)k}\}, \quad (96)$$

where the change from  $\alpha - 1/2$  to  $\alpha + 1/2$  in the exponent is due to a scaling factor. The inclusion (96) concludes the proof.  $\square$

## 7 Variations in the Orthogonal Direction.

In Lemma 2.1 we did a variation in the  $x_1$ -direction, that is in the direction tangential to the discontinuity set at the crack-tip, in order to derive estimates on the coefficient of the  $r^{1/2} \sin(\phi/2)$  term in the asymptotic expansion of a minimizing function  $u$ . In order to derive higher regularity of the crack-tip we need to make variations of the crack-tip in the  $x_2$ -direction, the direction orthogonal to the crack-tip. From these variations we will be able to derive another condition that minimizers satisfy at the crack-tip. The proof consists of a rather tedious calculation and is mostly interesting because *it works* and provides the result we desire.

The main proposition in this section is.

**Proposition 7.1.** *As in Proposition 3.1 we let  $(u^j, \Gamma_j)$  be a sequence of minimizers to the Mumford-Shah problem that are  $\epsilon_j$ -close to a crack tip for some sequence  $\epsilon_j \rightarrow 0$ . Then  $v^j = \frac{u^j - \Pi(u^j)}{\epsilon_j} \rightarrow v^0$ ,  $f_j \rightarrow f_0$  in the same sense as in Proposition 3.1. Then, by choosing the coordinate systems appropriately,*

$$v^0(r, \phi) = a + \sum_{k=2}^{\infty} a_k r^{\alpha_k} \cos(\alpha_k \phi) + \sum_{k=1}^{\infty} b_k r^{k-1/2} \sin((k-1/2)\phi) \quad (97)$$

and

$$f_0(x_1) = \sum_{k=2}^{\infty} 2a_k \sqrt{\frac{\pi}{2}} \sin(\alpha_k \pi) |x_1|^{\alpha_k + \frac{1}{2}}$$

The interesting thing with this proposition is that we sum the even terms of  $v^0$  from  $k = 2$  instead of  $k = 1$ . Thus the Proposition states that the cosine term of least oscillation does not appear in the linearization of minimizers.

The heart of the proof is the following Lemma.

**Lemma 7.1.** *Assume that  $(u, \Gamma_u)$  is a pair such that  $u \in W^{1,2}(B_1(0) \setminus \Gamma_u)$  and  $\Gamma_0$  is some connected rectifiable set connecting the origin to  $\partial B_1(0)$ .*

*Furthermore assume that, for some small  $\epsilon > 0$ ,*

$$u(x) = \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) + \epsilon r^{\alpha_1} \cos(\alpha_1 \phi) + o(\epsilon) \quad (98)$$

*and  $\Gamma_u \cap B_1(0) = \{(x_1, \epsilon f(x_1)); x_1 \in (-1, 0)\} \cap B_1(0)$  where*

$$f(x_1) = \sqrt{2\pi} \sin(\alpha_1 \pi) |x_1|^{\alpha_1+1/2} + o(\epsilon) = 2\sqrt{\frac{2}{\pi}} \frac{\alpha_1 \cos(\alpha_1 \pi)}{\alpha_1^2 - 1/4} |x_1|^{\alpha_1+1/2} + o(\epsilon), \quad (99)$$

*here  $o(\epsilon)$  is a  $W^{1,2}(B_1 \setminus \Gamma_u)$  function with norm  $\|o(\epsilon)\|_{W^{1,2}} \ll \epsilon$ .*

*Then  $(u, \Gamma_u)$  is not a minimizer to the Mumford-Shah problem.*

We will first prove Proposition 7.1 assuming Lemma 7.1 since the proof of the proposition is quite short and painless.

*Proof of Proposition 7.1:* By Lemma 6.1 we know that

$$v^0(r, \phi) = a + \sum_{k=1}^{\infty} a_k r^{\alpha_k} \cos(\alpha_k \phi) + \sum_{k=1}^{\infty} b_k r^{k-1/2} \sin((k-1/2)\phi) \quad (100)$$

and

$$f_0(x_1) = \sum_{k=1}^{\infty} a_k 2\sqrt{\frac{\pi}{2}} \sin(\alpha_k \pi) |x_1|^{\alpha_k + \frac{1}{2}}.$$

We need to show that  $a_1 = 0$  - we will therefore assume that  $a_1 \neq 0$  and derive a contradiction. If we rescale the functions  $u^j \mapsto u_s^j(x) = \frac{u^j(sx)}{\sqrt{s}}$  then the corresponding linearized sequence of functions  $v_s^j$  and  $f_s^j$  (corresponding to  $\Gamma_{u_s^j}$ ) will converge to

$$\begin{aligned} v_s^0(r, \phi) &= \sum_{k=1}^{\infty} a_k s^{\alpha_k-1/2} r^{\alpha_k} \cos(\alpha_k \phi) + \sum_{k=1}^{\infty} b_k r^{k-1/2} \sin((k-1/2)\phi) = \\ &= s^{\alpha_1-1/2} (a_1 r^{\alpha_1} \cos(\alpha_1 \phi) + O(s^{\alpha_2-\alpha_1})) \end{aligned} \quad (101)$$

and

$$\begin{aligned} f_0(x_1) &= \sum_{k=1}^{\infty} a_k s^{\alpha_k-1/2} 2\sqrt{\frac{\pi}{2}} \sin(\alpha_k \pi) |x_1|^{\alpha_k + \frac{1}{2}} = \\ &= s^{\alpha_1-1/2} \left( a_1 2\sqrt{\frac{\pi}{2}} \sin(\alpha_1 \pi) |x_1|^{\alpha_1 + \frac{1}{2}} + O(s^{\alpha_2-\alpha_1}) \right) \end{aligned}$$

where the  $O(s^{\alpha_2-\alpha_1})$  are to be understood as a  $W^{1,2}$ -function with norm controlled by  $Cs^{\alpha_2-\alpha_1}$  for some fixed  $C$ .

In particular we can deduce that

$$u_s^j(x) = \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) + \left( \epsilon_j s^{\alpha_1-1/2} \right) a_1 r^{\alpha_1} \cos(\alpha_k \phi) + \quad (102)$$

$$+ \left( \epsilon_j s^{\alpha_1-1/2} \right) O(s^{\alpha_2-\alpha_1}) + o(\epsilon_j).$$

If we introduce the notation  $\hat{\epsilon}_j = \epsilon_j s^{\alpha_1-1/2} a_1$  we may write (102)

$$u_s^j(x) = \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) + \hat{\epsilon}_j r^{\alpha_1} \cos(\alpha_k \phi) + \hat{\epsilon} O\left(\frac{s^{\alpha_2-\alpha_1}}{a_1}\right) + o(\hat{\epsilon}_j).$$

By choosing  $s$  small enough and  $j$  large enough we can conclude that

$$u_s^j(x) = \sqrt{\frac{2}{\pi}} r^{1/2} \sin(\phi/2) + \hat{\epsilon}_j a_1 r^{\alpha_1} \cos(\alpha_k \phi) + o(\hat{\epsilon}_j),$$

where  $o(\hat{\epsilon}_j)$  is, just as in Lemma 7.1, understood to be a  $W^{1,2}$ -function with norm  $\|o(\epsilon_j)\|_{W^{1,2}} < \epsilon$ .

Similarly, we may write

$$\Gamma_{u_s^j} = \{(x_1, \hat{\epsilon}_j f_s^j(x_1)); x_1 \in (-1, 0)\}$$

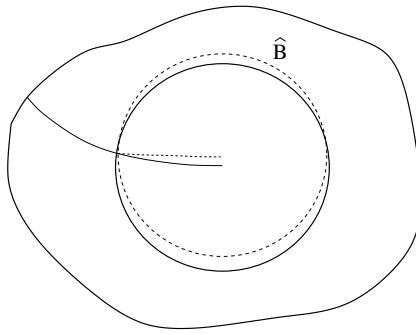
where

$$f_s^j(x_1) = \sqrt{2\pi} \sin(\alpha_1 \pi) r^{\alpha_1+1/2} + o(\hat{\epsilon}_j),$$

as in Lemma 7.1.

But then Lemma 7.1 implies that  $(u_s^j, \Gamma_{u_s^j})$  is not a minimizer which is a contradiction.  $\square$

*Proof of Lemma 7.1:* We will assume that  $(u, \Gamma_u)$  is a minimizer and satisfy (98) and (99) and then construct a pair  $(w, \Gamma_w)$  with less energy getting a contradiction. To that end we define a slightly shifted ball  $\hat{B} = B_1(\delta e_2)$  for some  $\epsilon^2 \ll |\delta| \ll \epsilon$  (dashed ball in the figure below).



**Figure 3:** The ball  $\hat{B}$  is a ball that is shifted by  $\delta$  in the  $x_2$  direction. The competing minimizer has its free discontinuity set  $\Gamma_w$  going from the boundary  $\partial \hat{B}$  to the center of the ball.

In  $\hat{B}$  we let  $(v, \Gamma_v)$  be the minimizer of the Mumford-Shah functional with boundary data  $u|_{\partial \hat{B}}$  and crack-tip at the point  $(0, \delta)$ . Clearly, since  $(v, \Gamma_v)$

minimizes the energy under the extra constraint that the crack-tip is at  $(0, \delta)$  and  $(u, \Gamma_u)$  is, by assumption, an unconstrained minimizer, we have that

$$J(u, \Gamma_u, \hat{B}) \leq J(v, \Gamma_v, \hat{B}). \quad (103)$$

In order to draw a contradiction from (103) we would have to calculate  $(v, \Gamma_v)$  which is rather complicated. Instead we will choose a comparison pair  $(w, \Gamma_w)$  such that  $w = v$  on  $\partial \hat{B}$  and  $\Gamma_w$  ends in  $(0, \delta)$ . Then, since  $(v, \Gamma_v)$  is a minimizer it follows that

$$J(u, \Gamma_u, \hat{B}) \leq J(v, \Gamma_v, \hat{B}) \leq J(w, \Gamma_w, \hat{B}). \quad (104)$$

In the calculations that follow we will center the coordinate system at the point  $(0, \delta)$  and denote the new coordinates by  $(\hat{x}_1, \hat{x}_2)$  and polar coordinates centered at  $(0, \delta)$  by  $(\hat{r}, \hat{\phi})$ .

Notice that if  $|\delta| < \epsilon$  then, by Taylor's Theorem, we have on  $\hat{B}$

$$\begin{aligned} u(\hat{x}_1, \hat{x}_2) &= u(x_1, x_2 + \delta) = u(x_1, x_2) + \delta \frac{\partial u(x_1, x_2)}{\partial x_2} + O(\delta^2) = \\ &= \sqrt{\frac{2}{\pi}} \sin(\phi/2) + \epsilon \cos(\alpha_1 \phi) + \delta \sqrt{\frac{1}{2\pi}} \cos(\phi/2) - \alpha_1 \epsilon \delta \sin((\alpha_1 - 1)\phi) + o(\epsilon). \end{aligned}$$

We may thus choose

$$\begin{aligned} w(\hat{r}, \hat{\phi}) &= \sqrt{\frac{2}{\pi}} \hat{r}^{1/2} \sin(\hat{\phi}/2) + \epsilon \hat{r}^{\alpha_1} \cos(\alpha_1 \hat{\phi}) + \\ &+ \delta \sqrt{\frac{1}{2\pi}} \hat{r}^{1/2} \cos(\hat{\phi}/2) - \alpha_1 \epsilon \delta \hat{r}^{\alpha_1-1} \sin((\alpha_1 - 1)\hat{\phi}) + o(\epsilon). \end{aligned} \quad (105)$$

In particular we choose  $w = v$  on  $\partial \hat{B}$ .

As the branch cut for  $w$ , or free discontinuity set,  $\Gamma_w$  we choose

$$\Gamma_w \cap \hat{B} = \{(\hat{x}_1, \epsilon f(\hat{x}_1) + \delta g(\hat{x}_1)) + \delta \hat{x}_1; \hat{x}_1 \in (-1, 0)\}.$$

Then, since we want  $\Gamma_w$  to end in the origin  $(\hat{x}_1, \hat{x}_2) = (0, 0)$  we must have  $g(0) = 0$ . We also want  $\Gamma_w \cap \partial \hat{B} = \Gamma_u \cap \partial \hat{B}$  we get a boundary condition on  $\partial \hat{B}$ . Using that  $\hat{x}_1 = -1 + O(\delta^2)$  this will lead to a boundary condition of the order  $g(-1) = O(\delta^2)$ . Since we will only calculate our asymptotic expansion to order  $\epsilon \delta$  it is no harm to treat  $g(x)$  as if  $g(x) = 0$ .

In order to derive a contradiction from (104) we need to calculate the relevant energies. It turns out that it is enough to calculate the energies up to order  $\epsilon \delta$ .

The rest of this proof consists of explicit calculations, mostly by using Taylor series, in order to show that

$$\begin{aligned} J(w, \Gamma_w, \hat{B}) - J(u, \Gamma_u, \hat{B}) &= \\ &= 2\epsilon \delta \sqrt{\frac{2}{\pi}} \frac{\alpha_1^2 + 1/4}{\alpha_1^2 - 1/4} \cos(\alpha_1 \pi) + o(\epsilon \delta). \end{aligned}$$

Choosing the right sign of  $\delta$  this implies that  $J(w, \Gamma_w, \hat{B}) < J(u, \Gamma_u, \hat{B})$  which contradicts the minimality assumption of  $(u, \Gamma_u)$ .



**Claim 1.** *The difference in of the Dirichlet energies of  $u$  and  $w$  are:*

$$\int_{\hat{B} \setminus \Gamma_w} |\nabla w|^2 - \int_{\hat{B} \setminus \Gamma_u} |\nabla u|^2 = 2\epsilon\delta \sqrt{\frac{2}{\pi}} \frac{\alpha_1 - 1/2}{\alpha_1 + 1/2} \cos(\alpha_1\pi) + o(\epsilon\delta)$$

*Proof of claim 1:* We start by calculating the Dirichlet energy of  $u$ :

$$\int_{\hat{B} \setminus \Gamma_u} |\nabla u|^2 = \int_{B_1(0) \setminus \Gamma_u} |\nabla u|^2 + \delta \frac{d}{dt} \int_{B_1(te_2) \setminus \Gamma_u} |\nabla u|^2 + O(\delta^2). \quad (106)$$

Using the assumption (98) we see that

$$\int_{B_1(0) \setminus \Gamma_u} |\nabla u|^2 = \int_{B_1(0) \setminus \Gamma_u} \frac{2}{\pi} |\nabla(r^{1/2} \sin(\phi/2))|^2 + \quad (107)$$

$$+ \int_{B_1(0) \setminus \Gamma_u} 2\sqrt{\frac{2}{\pi}} \epsilon \nabla(r^{1/2} \sin(\phi/2)) \cdot \nabla(r^{\alpha_1} \cos(\alpha_1\phi)) + \quad (108)$$

$$+ \int_{B_1(0) \setminus \Gamma_u} \epsilon^2 |\nabla(r^{\alpha_1} \cos(\alpha_1\phi))|^2 + 2\sqrt{\frac{2}{\pi}} \nabla(r^{1/2} \sin(\phi/2)) \cdot \nabla o(\epsilon) + \quad (109)$$

$$+ \int_{B_1(0) \setminus \Gamma_u} 2\epsilon \nabla(r^{\alpha_1} \cos(\alpha_1\phi)) \cdot \nabla o(\epsilon) + |\nabla o(\epsilon)|^2 = \sum_{j=1}^6 I_{u,j}. \quad (110)$$

Since  $e_2 \cdot \nu = \sin(\phi)$  on  $\partial B_1(0)$  where  $\nu$  is the normal of  $B_1(0)$  we can calculate that

$$\delta \frac{d}{dt} \int_{B_1(0) \setminus \Gamma_u} |\nabla u|^2 = \delta \int_{\partial B_1(0) \setminus \Gamma_u} \sin(\phi) \frac{2}{\pi} |\nabla(r^{1/2} \sin(\phi/2))|^2 + \quad (111)$$

$$+ \delta \int_{\partial B_1(0) \setminus \Gamma_u} 2\sqrt{\frac{2}{\pi}} \epsilon \sin(\phi) \nabla(r^{1/2} \sin(\phi/2)) \cdot \nabla(r^{\alpha_1} \cos(\alpha_1\phi)) + \quad (112)$$

$$+ \delta \int_{\partial B_1(0) \setminus \Gamma_u} \epsilon^2 \sin(\phi) |\nabla(r^{\alpha_1} \cos(\alpha_1\phi))|^2 + 2\sqrt{\frac{2}{\pi}} \sin(\phi) \nabla(r^{1/2} \sin(\phi/2)) \cdot \nabla o(\epsilon) + \quad (113)$$

$$+ 2\delta \int_{\partial B_1(0) \setminus \Gamma_u} \sqrt{\frac{2}{\pi}} \sin(\phi) \nabla(r^{1/2} \sin(\phi/2)) \cdot \nabla o(\epsilon) + \quad (114)$$

$$+ \delta \int_{\partial B_1(0) \setminus \Gamma_u} 2\epsilon \sin(\phi) \nabla(r^{\alpha_1} \cos(\alpha_1\phi)) \cdot \nabla o(\epsilon) + \sin(\phi) |\nabla o(\epsilon)|^2 = \quad (115)$$

$$= I_{u,\partial,1} + I_{u,\partial,2} + I_{u,\partial,3} + I_{u,\partial,4} + I_{u,\partial,5} + I_{u,\partial,6}. \quad (116)$$

We begin by estimating  $I_{u,\partial,1}, I_{u,\partial,2}, \dots, I_{u,\partial,6}$ . Noticing that  $I_{u,\partial,1}, I_{u,\partial,3}$  and  $I_{u,\partial,6}$  have odd integrands integrated over  $\int_{-\pi-c\epsilon}^{\pi-c\epsilon}$  their contribution will be of order  $o(\epsilon\delta)$ . The integrals  $I_{u,\partial,4}$  and  $I_{u,\partial,5}$  are of order  $o(\epsilon\delta)$ . We can thus conclude that

$$\delta \frac{d}{dt} \int_{B_1(0) \setminus \Gamma_u} |\nabla u|^2 = I_{u,\partial,2} + o(\epsilon\delta) \quad (117)$$

$$= \epsilon\delta 2\sqrt{\frac{2}{\pi}} \int_{\partial B_1(0) \setminus \Gamma_u} \sin(\phi) \nabla(r^{1/2} \sin(\phi/2)) \cdot \nabla(r^{\alpha_1} \cos(\alpha_1\phi)) + o(\epsilon\delta) \quad (118)$$

Using our choice of  $w$  in (105) we can calculate

$$\int_{\hat{B} \setminus \Gamma_w} |\nabla w|^2 = \int_{\hat{B} \setminus \Gamma_w} \frac{2}{\pi} |\nabla(\hat{r}^{1/2} \sin(\hat{\phi}/2))|^2 + \quad (119)$$

$$+ \int_{\hat{B} \setminus \Gamma_w} \delta \sqrt{\frac{2}{\pi}} \nabla(\hat{r}^{1/2} \sin(\hat{\phi}/2)) \cdot \nabla(\hat{r}^{1/2} \cos(\hat{\phi}/2)) + \quad (120)$$

$$+ \int_{\hat{B} \setminus \Gamma_w} \frac{\delta^2}{2\pi} |\nabla(\hat{r}^{1/2} \cos(\hat{\phi}/2))|^2 - \quad (121)$$

$$- \int_{\hat{B} \setminus \Gamma_w} 2\epsilon\delta \sqrt{\frac{2}{\pi}} \alpha_1 \nabla(\hat{r}^{1/2} \sin(\hat{\phi}/2)) \cdot \nabla(\hat{r}^{\alpha_1-1} \sin((\alpha_1-1)\hat{\phi})) - \quad (122)$$

$$- \int_{\hat{B} \setminus \Gamma_w} \epsilon\delta^2 \alpha_1 \nabla(\hat{r}^{1/2} \cos(\hat{\phi}/2)) \cdot \nabla(\hat{r}^{\alpha_1-1} \sin((\alpha_1-1)\hat{\phi})) + \quad (123)$$

$$+ \int_{\hat{B} \setminus \Gamma_w} \epsilon^2 \delta^2 \alpha_1^2 |\nabla(\hat{r}^{\alpha_1-1} \sin((\alpha_1-1)\hat{\phi}))|^2 - \quad (124)$$

$$- \int_{\hat{B} \setminus \Gamma_w} 2\alpha_1 \epsilon^2 \delta \nabla(\hat{r}^{\alpha_1-1} \sin((\alpha_1-1)\hat{\phi})) \cdot \nabla(\hat{r}^{\alpha_1} \cos(\alpha_1 \hat{\phi})) + \quad (125)$$

$$+ \int_{\hat{B} \setminus \Gamma_w} \epsilon^2 |\nabla(\hat{r}^{\alpha_1} \cos(\alpha_1 \hat{\phi}))|^2 + 2\sqrt{\frac{2}{\pi}} \nabla(\hat{r}^{1/2} \sin(\hat{\phi}/2)) \cdot \nabla o(\epsilon) - \quad (126)$$

$$- \int_{\hat{B} \setminus \Gamma_w} 2\alpha_1 \epsilon \delta \nabla(\hat{r}^{\alpha_1-1} \sin((\alpha_1-1)\hat{\phi})) \cdot \nabla o(\epsilon) + \quad (127)$$

$$+ \int_{\hat{B} \setminus \Gamma_w} 2\epsilon \nabla(\hat{r}^{\alpha_1} \cos(\alpha_1 \hat{\phi})) \cdot \nabla o(\epsilon) + |\nabla o(\epsilon)|^2 + \quad (128)$$

$$+ \int_{\hat{B} \setminus \Gamma_w} 2\epsilon \sqrt{\frac{2}{\pi}} \nabla(\hat{r}^{1/2} \sin(\hat{\phi}/2)) \cdot \nabla(\hat{r}^{\alpha_1} \cos(\alpha_1 \hat{\phi})) + \quad (129)$$

$$+ \int_{\hat{B} \setminus \Gamma_w} \epsilon \delta \sqrt{\frac{2}{\pi}} \nabla(\hat{r}^{1/2} \cos(\hat{\phi}/2)) \cdot \nabla(\hat{r}^{\alpha_1} \cos(\alpha_1 \hat{\phi})) + \quad (130)$$

$$+ \int_{\hat{B} \setminus \Gamma_w} \delta \sqrt{\frac{2}{\pi}} \nabla(\hat{r}^{1/2} \cos(\hat{\phi}/2)) \cdot \nabla o(\epsilon) = \sum_{j=1}^{15} \hat{I}_{w,j}. \quad (131)$$

Our next goal is to investigate which of the integrals  $\hat{I}_{w,1}, \dots, \hat{I}_{w,15}$  that are of order  $\epsilon\delta$  or higher. Clearly  $\hat{I}_{w,3}, \hat{I}_{w,5}, \hat{I}_{w,6}, \hat{I}_{w,7}, \hat{I}_{w,10}$  and  $\hat{I}_{w,15}$  are of order  $o(\epsilon\delta)$  and can thus be disregarded.

We also notice that we may change the domain of integration and variables of integration from  $\hat{B}$ ,  $\hat{r}$  and  $\hat{\phi}$  to  $B_1(0)$ ,  $r$  and  $\phi$  without changing the value of the integral and thus conclude that

$$\int_{\hat{B} \setminus \Gamma_w} |\nabla w|^2 = \int_{B_1(0) \setminus \Gamma_w} \frac{2}{\pi} |\nabla(r^{1/2} \sin(\phi/2))|^2 + \quad (132)$$

$$+ \int_{B_1(0) \setminus \Gamma_w} \delta \sqrt{\frac{2}{\pi}} \nabla(r^{1/2} \sin(\phi/2)) \cdot \nabla(r^{1/2} \cos(\phi/2)) - \quad (133)$$

$$- \int_{B_1(0) \setminus \Gamma_w} 2\epsilon\delta\sqrt{\frac{2}{\pi}}\alpha_1\nabla(r^{1/2}\sin(\phi/2)) \cdot \nabla(r^{\alpha_1-1}\sin((\alpha_1-1)\phi)) + \quad (134)$$

$$+ \int_{B_1(0) \setminus \Gamma_w} \epsilon^2|\nabla(r^{\alpha_1}\cos(\alpha_1\phi))|^2 + 2\sqrt{\frac{2}{\pi}}\nabla(r^{1/2}\sin(\phi/2)) \cdot \nabla o(\epsilon) + \quad (135)$$

$$+ \int_{B_1(0) \setminus \Gamma_w} 2\epsilon\sqrt{\frac{2}{\pi}}\nabla(r^{1/2}\sin(\phi/2)) \cdot \nabla(r^{\alpha_1}\cos(\alpha_1\phi)) + \quad (136)$$

$$+ \int_{\hat{B} \setminus \Gamma_w} 2\epsilon\nabla(r^{\alpha_1}\cos(\alpha_1\phi)) \cdot \nabla o(\epsilon) + |\nabla o(\epsilon)|^2 + \quad (137)$$

$$+ \int_{B_1(0) \setminus \Gamma_w} \epsilon\delta\sqrt{\frac{2}{\pi}}\nabla(r^{1/2}\cos(\phi/2)) \cdot \nabla(r^{\alpha_1}\cos(\alpha_1\phi)) + o(\epsilon\delta) = \quad (138)$$

$$= \sum_{j=1}^8 I_{w,j} + o(\epsilon\delta). \quad (139)$$

Next we notice that the only difference between  $I_{w,1}, I_{w,4}, I_{w,8}$  and  $I_{u,1}, I_{u,3}, I_{u,6}$  is the branch-cut where we place the discontinuity - this does not affect the value of the integrals. Moreover  $I_{w,5}, I_{w,7}$  have the same integrands as  $I_{u,4}, I_{u,5}$  and the difference in the branch-cut only affects the domain of integration of a set of area  $\delta$ . Since all the integrands in  $I_{w,5}, I_{w,7}, I_{u,4}$  and  $I_{u,5}$  are of order  $o(\epsilon)$  we can conclude that the difference between these integrals are of order  $o(\epsilon\delta)$ .

We can thus conclude that the difference in Dirichlet energy is

$$\begin{aligned} \int_{\hat{B} \setminus \Gamma_w} |\nabla w|^2 - \int_{\hat{B} \setminus \Gamma_u} |\nabla u|^2 &= \sum_{j=1}^8 I_{w,j} - \sum_{j=1}^6 I_{u,j} - I_{u,\partial,2} + o(\epsilon\delta) = \\ &= I_{w,2} + I_{w,3} + I_{w,6} + I_{w,9} - I_{u,2} - I_{u,\partial,2} = \\ &= \int_{B_1(0) \setminus \Gamma_w} \delta\sqrt{\frac{2}{\pi}}\nabla(r^{1/2}\sin(\phi/2)) \cdot \nabla(r^{1/2}\cos(\phi/2)) - \end{aligned} \quad (140)$$

$$- \int_{B_1(0) \setminus \Gamma_w} 2\epsilon\delta\sqrt{\frac{2}{\pi}}\alpha_1\nabla(r^{1/2}\sin(\phi/2)) \cdot \nabla(r^{\alpha_1-1}\sin((\alpha_1-1)\phi)) + \quad (141)$$

$$+ \int_{B_1(0) \setminus \Gamma_w} - \int_{B_1(0) \setminus \Gamma_u} \left[ 2\epsilon\sqrt{\frac{2}{\pi}}\nabla(r^{1/2}\sin(\phi/2)) \cdot \nabla(r^{\alpha_1}\cos(\alpha_1\phi)) \right] + \quad (142)$$

$$+ \int_{B_1(0) \setminus \Gamma_w} \epsilon\delta\sqrt{\frac{2}{\pi}}\nabla(r^{1/2}\cos(\phi/2)) \cdot \nabla(r^{\alpha_1}\cos(\alpha_1\phi)) - \quad (143)$$

$$- \epsilon\delta 2\sqrt{\frac{2}{\pi}} \int_{\partial B_1(0) \setminus \Gamma_u} \sin(\phi)\nabla(r^{1/2}\sin(\phi/2)) \cdot \nabla(r^{\alpha_1}\cos(\alpha_1\phi)) + o(\epsilon\delta) = \quad (144)$$

$$= \sum_{j=1}^5 J_j. \quad (145)$$

We need to estimate these integrals one by one. We start with

$$J_1 = \int_{B_1(0) \setminus \Gamma_w} \delta\sqrt{\frac{2}{\pi}}\nabla(r^{1/2}\sin(\phi/2)) \cdot \nabla(r^{1/2}\cos(\phi/2)) = 0,$$

since  $\nabla(r^{1/2} \sin(\phi/2))$  and  $\nabla(r^{1/2} \cos(\phi/2))$  are orthogonal on the disk and their values are independent on where we put the branch-cut.

To estimate  $J_2$  we proceed as follows. We start by noticing that if we integrate over  $B_1(0) \setminus \{x_2 = 0, x_1 < 0\}$  then we change the area of integration by a set of measure of the order  $\epsilon$  and we may thus estimate

$$\begin{aligned}
J_2 &= - \int_{B_1(0) \setminus \Gamma_w} 2\epsilon\delta \sqrt{\frac{2}{\pi}} \alpha_1 \nabla(r^{1/2} \sin(\phi/2)) \cdot \nabla(r^{\alpha_1-1} \sin((\alpha_1-1)\phi)) = \\
&= - \int_{B_1(0) \setminus \Gamma_0} 2\epsilon\delta \sqrt{\frac{2}{\pi}} \alpha_1 \nabla(r^{1/2} \sin(\phi/2)) \cdot \nabla(r^{\alpha_1-1} \sin((\alpha_1-1)\phi)) + o(\epsilon\delta) = \\
&= \epsilon\delta \sqrt{\frac{2}{\pi}} \alpha_1 \int_{\partial B_1(0) \setminus \Gamma_0} \sin(\phi/2) \sin((\alpha_1-1)\phi) + o(\epsilon\delta) = \\
&= \epsilon\delta \sqrt{\frac{2}{\pi}} \alpha_1 \int_{\partial B_1(0) \setminus \Gamma_0} \frac{1}{2} (\cos((\alpha_1-3/2)\phi) - \cos((\alpha_1-1/2)\phi)) + o(\epsilon\delta) = \\
&= \epsilon\delta \sqrt{\frac{2}{\pi}} \alpha_1 \left( \frac{\sin((\alpha_1-3/2)\pi)}{\alpha_1-3/2} - \frac{\sin((\alpha_1-1/2)\pi)}{\alpha_1-1/2} \right) + o(\epsilon\delta) = \\
&= c_2 \epsilon\delta + o(\epsilon\delta). \tag{146}
\end{aligned}$$

where we used an integration by parts and that  $\frac{\partial \sin(\phi/2)}{\partial \nu_{\pm}} = 0$  on the part of the boundary given by  $\{x_2 = 0, x_1 < 0\}$  in the second step of the calculation. We also define the constant  $c_2$  in the final equality.

To estimate  $J_3$  use an integration by parts and transfer the integrals to  $\Gamma_w$  and  $\Gamma_u$ . We also approximate, up to (the lower) order  $\epsilon$ , the normal at  $\Gamma_w$  and  $\Gamma_u$  by  $\pm e_2$  and notice that the boundary integral on  $\partial B_1(0)$  cancels. Furthermore we use that the angle  $\phi$  on  $\Gamma_w$  and  $\Gamma_u$  is given by  $\pm\pi - \arctan(\epsilon f/r + \delta g/r - \delta) \approx \pm\pi + \epsilon f(-r)/r + \delta g(-r)/r - \delta$  and  $\pm\pi - \arctan(\epsilon f/r) \approx \pm\pi + \epsilon f/r$  - estimates that holds for  $r \in (0, 1)$  since we may assume that  $\frac{g(-r)}{r} \rightarrow 0$  as  $r \rightarrow 0$ .

Our estimate of  $J_3$  then becomes

$$\begin{aligned}
J_3 &= 2\epsilon \sqrt{\frac{2}{\pi}} \int_{B_1(0) \setminus \Gamma_w} - \int_{B_1(0) \setminus \Gamma_u} \left[ \nabla(r^{1/2} \sin(\phi/2)) \cdot \nabla(r^{\alpha_1} \cos(\alpha_1 \phi)) \right] = \\
&= 2\epsilon \sqrt{\frac{2}{\pi}} \int_0^1 \frac{r^{\alpha_1-1/2}}{2} \cos\left(\frac{\pi - \epsilon f(-r)/r + \delta}{2}\right) \times \tag{147}
\end{aligned}$$

$$\begin{aligned}
&\times \cos(\alpha_1 (\pi - \epsilon f(-r)/r + \delta)) - \\
&- 2\epsilon \sqrt{\frac{2}{\pi}} \int_0^1 \frac{r^{\alpha_1-1/2}}{2} \cos\left(\frac{\pi - \epsilon f(-r)/r}{2}\right) \cos(\alpha_1 (\pi - \epsilon f(-r)/r)) + \tag{148}
\end{aligned}$$

$$\begin{aligned}
&- 2\epsilon \sqrt{\frac{2}{\pi}} \int_0^1 \frac{r^{\alpha_1-1/2}}{2} \cos\left(\frac{\pi + \epsilon f(-r)/r - \delta}{2}\right) \times \tag{149} \\
&\times \cos(\alpha_1 (\pi + \epsilon f(-r)/r - \delta)) -
\end{aligned}$$

$$\begin{aligned}
&+ 2\epsilon \sqrt{\frac{2}{\pi}} \int_0^1 \frac{r^{\alpha_1-1/2}}{2} \cos\left(\frac{\pi + \epsilon f(-r)/r}{2}\right) \cos(\alpha_1 (\pi + \epsilon f(-r)/r)) + o(\epsilon\delta). \tag{150}
\end{aligned}$$

Notice that we may estimate the first two integrands, (147) and (148), by a Taylor expansion:

$$\begin{aligned}
& \cos\left(\frac{\pi - \epsilon f(-r)/r + \delta}{2}\right) \cos(\alpha_1 (\pi - \epsilon f(-r)/r + \delta)) - \\
& - \cos\left(\frac{\pi - \epsilon f(-r)/r}{2}\right) \cos(\alpha_1 (\pi - \epsilon f(-r)/r)) = \\
& = \frac{\delta}{2} \frac{\partial \left( \cos\left(\frac{\pi - \epsilon f(-r)/r}{2} + t\right) \cos(\alpha_1 (\pi - \epsilon f(-r)/r + t)) \right)}{\partial t} \Big|_{t=0} = \\
& = \frac{\delta}{2} (\cos(\alpha_1 \pi) + O(\epsilon_j)).
\end{aligned}$$

Taking into consideration that we multiply the integrals by  $\epsilon$  the contribution of (147) and (148) is of order

$$- \epsilon \delta \frac{1}{2} \sqrt{\frac{2}{\pi}} \cos(\alpha_1 \pi) \int_0^1 r^{\alpha_1 - 1/2} dr + o(\epsilon \delta). \quad (151)$$

Similarly, one can calculate the contribution of (149) and (150) to  $J_3$  to be

$$- \epsilon \delta \frac{1}{2} \sqrt{\frac{2}{\pi}} \cos(\alpha_1 \pi) \int_0^1 r^{\alpha_1 - 1/2} dr. \quad (152)$$

We conclude that

$$\begin{aligned}
J_3 &= -\epsilon \delta \sqrt{\frac{2}{\pi}} \cos(\alpha_1 \pi) \int_0^1 r^{\alpha_1 - 1/2} dr + o(\epsilon \delta) + o(\epsilon \delta) = \\
&= -\epsilon \delta \sqrt{\frac{2}{\pi}} \frac{\cos(\alpha_1 \pi)}{\alpha_1 + 1/2} + o(\epsilon \delta).
\end{aligned}$$

Next we move on to  $J_4$  which we estimate in a similar way as we estimated  $J_2$ . In particular:

$$\begin{aligned}
J_4 &= \epsilon \delta \sqrt{\frac{2}{\pi}} \int_{B_1(0) \setminus \{x_2=0, x_1 < 0\}} \nabla(r^{1/2} \cos(\phi/2)) \cdot \nabla(r^{\alpha_1} \cos(\alpha_1 \phi)) + o(\epsilon \delta) = \\
&= \epsilon \delta \sqrt{\frac{2}{\pi}} \int_{\partial B_1(0) \setminus \{x_2=0, x_1 < 0\}} \alpha_1 \cos(\phi/2) \cos(\alpha_1 \phi) + o(\epsilon \delta) = \\
&= \epsilon \delta \sqrt{\frac{2}{\pi}} \alpha_1 \left( \frac{\sin(\alpha_1 + 1/2)\pi}{\alpha_1 + 1/2} + \frac{\sin((\alpha_1 - 1/2)\pi)}{\alpha_1 - 1/2} \right) + o(\epsilon \delta) = \\
&= c_4 \epsilon \delta + o(\epsilon \delta),
\end{aligned} \quad (153)$$

where  $c_4$  is defined by the last equality (153).

And, finally, to estimate  $J_5$  we calculate

$$J_5 = -\epsilon \delta 2 \sqrt{\frac{2}{\pi}} \int_{\partial B_1(0) \setminus \Gamma_u} \sin(\phi) \nabla(r^{1/2} \sin(\phi/2)) \cdot \nabla(r^{\alpha_1} \cos(\alpha_1 \phi)) =$$

$$\begin{aligned}
&= -2\epsilon\delta\sqrt{\frac{2}{\pi}}\alpha_1\frac{\cos(\alpha_1\pi)}{(\alpha_1 - 3/2)(\alpha_1 + 1/2)} + o(\epsilon\delta) = \\
&= c_5\epsilon\delta + o(\epsilon\delta).
\end{aligned} \tag{154}$$

We have thus proved that the difference in Dirichlet energy of  $w$  and  $u$  is

$$\begin{aligned}
&\int_{\hat{B}\setminus\Gamma_w} |\nabla w|^2 - \int_{\hat{B}\setminus\Gamma_u} |\nabla u|^2 = (c_2 + c_4 + c_5)\epsilon\delta + \\
&\quad -\epsilon\delta\sqrt{\frac{2}{\pi}}\frac{\cos(\alpha_1\pi)}{\alpha_1 + 1/2} + o(\epsilon\delta) = \\
&= \epsilon\delta\sqrt{\frac{2}{\pi}}\left(2\alpha_1\frac{\cos(\alpha_1\pi)}{\alpha_1 + 1/2} - \frac{\cos(\alpha_1\pi)}{\alpha_1 + 1/2}\right) + o(\epsilon\delta) = \\
&= 2\epsilon\delta\sqrt{\frac{2}{\pi}}\frac{\alpha_1 - 1/2}{\alpha_1 + 1/2}\cos(\alpha_1\pi) + o(\epsilon\delta)
\end{aligned} \tag{155}$$

where  $c_2, c_4$  and  $c_5$  are the explicit constants in (146), (153) and (154). The final three equalities after (155) is just a simplification.

**Claim 2:** *The difference in Hausdorff measure between  $\Gamma_u$  and  $\Gamma_w$  is*

$$\mathcal{H}^1(\Gamma_w) - \mathcal{H}^1(\Gamma_u) = 2\epsilon\delta\sqrt{\frac{2}{\pi}}\frac{\alpha_1\cos(\alpha_1\pi)}{\alpha_1^2 - 1/4} + o(\epsilon\delta).$$

*Proof of Claim 2:* Fortunately, it is very easy to calculate the difference in Hausdorff measure:

$$\begin{aligned}
\mathcal{H}^1(\Gamma_w) - \mathcal{H}^1(\Gamma_u) &= \int_0^1 \sqrt{1 + |\epsilon f'(-r) - \delta|^2} dr - \int_0^1 \sqrt{1 + |\epsilon f'(-r)|^2} dr = \\
&= \int_0^1 \frac{-\epsilon\delta f'(-r)}{\sqrt{1 + |\epsilon_j f'_j(-r)|^2}} dr + O(\delta^2) = 2\epsilon\delta\sqrt{\frac{2}{\pi}}\frac{\alpha_1\cos(\alpha_1\pi)}{\alpha_1^2 - 1/4} + o(\epsilon\delta),
\end{aligned} \tag{156}$$

where we inserted the explicit expression of  $f$ ,

$$f'(x_1) = 2\sqrt{\frac{2}{\pi}}\alpha_1\frac{\cos(\alpha_1\pi)}{\alpha_1 - 1/2}r^{\alpha_1-1/2} + o(\epsilon),$$

in the final equality. We also used the approximation

$$\sqrt{1 + |\epsilon_j f'_j(-r)|^2} = 1 + O(\epsilon_j^2)$$

in the final intquality (156).

With Claim 1 and Claim 2 we are able to write down the difference in energy of  $(w, \Gamma_w)$  and  $(u, \Gamma_u)$ :

$$\begin{aligned}
&J(w, \Gamma_w, \hat{B}) - J(u, \Gamma_u, \hat{B}) = \\
&2\epsilon\delta\sqrt{\frac{2}{\pi}}\frac{\alpha_1 - 1/2}{\alpha_1 + 1/2}\cos(\alpha_1\pi) + 2\epsilon\delta\sqrt{\frac{2}{\pi}}\frac{\alpha_1\cos(\alpha_1\pi)}{\alpha_1^2 - 1/4} + o(\epsilon\delta) =
\end{aligned}$$

$$= 2\epsilon\delta\sqrt{\frac{2}{\pi}\frac{\alpha_1^2+1/4}{\alpha_1^2-1/4}}\cos(\alpha_1\pi) + o(\epsilon\delta).$$

By choosing the right sign on  $\delta$  we may make the last expression less than zero which contradicts our assumption that  $\epsilon \neq 0$ . This finishes the proof.  $\square$

**Theorem 7.1.** *There exists an  $\epsilon_0 > 0$  such that if  $(u, \Gamma)$  is  $\epsilon$ -close to a crack-tip solution for some  $\epsilon \leq \epsilon_0$  then  $\Gamma_u$  is  $C^{2,\alpha}$  at the crack-tip for every  $\alpha < \alpha_2 - 3/2$ ,  $\alpha_2 \approx 2.099$ .*

*This in the sense that the tangent at the crack-tip is a well defined line, which we may assume to be  $\{(x_1, 0); x_1 \in \mathbb{R}\}$ , and there exists a constant  $C_\alpha$  such that*

$$\Gamma_u \subset \{(x_1, x_2); |x_2| < C_\alpha \epsilon |x_1|^{2+\alpha}, x_1 < 0\}.$$

*Here  $C_\alpha$  may depend on  $\alpha$  but not on  $\epsilon < \epsilon_0$ .*

*Proof:* The proof is line for line the same at the proof of Corollary 6.1.  $\square$

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